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ABSTRACT

There are a number of recurring topics in the articles that comprise this yearbook, such as the nature of both informal and demonstrative geometry, the reasons for teaching both, and the extent of such courses. Other emphasized topics are use of the analytic method, whether to combine plane and solid geometry, the place of algebra and trigonometry in a geometry course, and geometry in the junior high school. There is a chapter by Birkhoff and Beatley explaining a "number-based" approach to geometry. Included is a listing of fundamental principles, basic theorems, and a discussion of the advantages of this approach. Another author argues that transfer of training is a desirable learning outcome and that the study of geometry is well-suited for this. Other individual chapters deal with using graphs in the study of congruences, the use of indirect proof, symmetry, and a guide for evaluating geometry textbooks. (LS)

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# THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

## THE FIFTH YEARBOOK

# THE TEACHING OF GEOMETRY

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# EDITOR'S PREFACE

Four Yearbooks preceding this one have been published by the National Council of Teachers of Mathematics. The first dealt with "A Survey of Progress in the Past Twenty-five Years," the second with "Curriculum Problems in Teaching Mathematics," the third with "Selected Topics in the Teaching of Mathematics," and the fourth with "Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910." These Yearbooks have set a standard not only in quality and in appearance but in cost to those who are interested in reading them. That the Yearbooks have been well received is evidenced by the fact that the first one has been out of print for two years, only a few copies of the second are available, and there is only a limited number left of the third and fourth.

The fifth Yearbook is intended to supplement and assist the National Committee recently appointed by the Mathematical Association of America and the National Council of Teachers of Mathematics in studying the feasibility of a combined one-year course in plane and solid geometry. However, this Yearbook is not in any sense a report of this committee. It is intended mainly to stimulate discussion among teachers of mathematics throughout the country and thus to clarify the situation for the committee as much as possible.

The reason for having two different chapters on "A Unit of Demonstrative Geometry for the Ninth Year" is that we wish to have different points of view on this somewhat new departure.

I wish to express my personal appreciation as well as that of the National Council to all the contributors who have helped so much to make this Yearbook possible.

W. D. Reeve.

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# THE TEACHING OF GEOMETRY

# THE TEACHING OF GEOMETRY

By WILLIAM DAVID REEVE

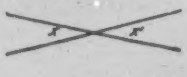
*Teachers College, Columbia University, New York City*

**What "Geometry" Means.** The word "geometry" originally meant "earth measure." That is, geometry was at first thought of as we think of surveying at the present time. This original meaning of the word, however, was abandoned many centuries ago and "geometry" came to be used to designate that part of mathematics dealing with points, lines, surfaces, and solids or with some combination of these geometric magnitudes.

## I. Two Types of Geometry

**Informal Geometry.** At the present time we recognize two types of geometry. The first type is commonly referred to as "intuitive geometry," and the second as "demonstrative geometry." The name "intuitive geometry" is inadequate inasmuch as it does not include that part of geometry which we often refer to as "experimental geometry." The fact is that the name "intuitive" has been used largely because it seemed to be the best one available. For pedagogical purposes it would improve the situation greatly and would make for clearness if we could adopt the term "informal geometry" in the sense that no formal demonstrations are to be given. This term would include the other two and we could then treat the subject in the following manner:

1. *Intuitive Geometry.* Here the child looks at a figure and says that certain things are so because he thinks they could not be otherwise. For example, he looks at two intersecting lines and says that the vertical angles $x$ and $x'$ thus formed are equal because "he feels it in his bones." His intuition tells him that it is true, and in such a case, experiment or proof, informal or otherwise, will not be necessary to convince him that his intuition is correct. In fact, if he measures a pair of vertical angles, he may not be able to get the same number of degrees in each.



Poincaré denied that certainty can come from intuition, but he gave it a prominent place in the teaching of mathematics. He said:

> I have already had occasion to insist on the place intuition should hold in the teaching of the mathematical sciences. Without it young minds could not make a beginning in the understanding of mathematics; they could not learn to love it and would see in it only a vain logomachy; above all, without intuition they would never become capable of applying mathematics.

The tendency of the modern pure mathematician, especially since the critical investigation of the foundations of Euclid (particularly his parallel postulate), has been to reduce the use of intuition to a minimum. Since the subject of geometry has been considered a commingling of intuition, "the instrument of invention," and logic, "the instrument of demonstration," with the latter predominating, it seems safe to give each an important rôle in teaching geometry.

Carson[1] defines intuitions as "merely a particular class of assumptions or postulates, such as form the basis of every science." He says:

> It is this type of exercise in drawing and measurement which I regard as an attack upon intuition. It replaces this natural and inevitable process by hasty generalization from experiments of the crudest type. Some advocates of these exercises defend them on the ground that they lead to the formation of intuitions, and that the pupils were not previously cognisant of the facts involved. But in the first place, a conscious induction from deliberate experiments is not an intuition; it lacks each of the special elements connoted by the term.[2]

2. *Experimental Geometry.* Here, by means of a simple experiment of one kind or another, the child discovers the truth of some statement. For example, a small child playing with a toy horse soon discovers that the horse is more stable on three legs than on four. Or by cutting out a paper triangle, tearing off the three angles and placing them together (adjacent), the pupil may discover that their sum is equal to a half turn of a rotating line (180°). These experiments prove nothing in the deductive sense, but they bring about conviction in the mind of the child and that is what

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[1] Carson, G. St. L., *Essays on Mathematical Education.* Chapter on "Intuition," p. 21. Ginn and Co., 1913.

[2] *Ibid.*, p. 27.

we want at this early stage where the "wonder motive" is the guiding motive in learning.

It is clear that some teachers will say that intuition never does anything more than furnish the provisional hypotheses which must subsequently be investigated for their truth or falsity and that this can be done by various methods which differ in their scientific exactness. Such a view would not consider experimental geometry as I have discussed it a separate type of informal geometry, but merely one of the modes of testing provisional hypotheses. From a pedagogical point of view, however, it may prove useful to adopt the scheme I have described. It is clear that the child who readily concludes that vertical angles formed by two intersecting lines are equal will not similarly conclude that the sum of the angles of a triangle is 180 degrees. Both are provisional hypotheses, to be sure, but the first is more or less certain, while the second demands some kind of experiment. Here is where paper-folding, models, measurement, and the like, come in when one is teaching geometry informally or formally. Usually the pupils agree very well as to when so-called formal proofs (mental experiments) are necessary.

**Content of Informal Geometry.** If we consider any definite object, we see that, aside from such matters as its structure and its purpose, there are three questions of a geometric nature that can be asked concerning it. They are: (1) What is the shape of the object? (2) What is its size? and (3) Where is it? These questions represent the initial stages in the study of geometry; they seek for the probable facts of the case. The first of these questions is probably the one that would be asked first by a primitive people and it gives rise to what we call *The Geometry of Form;* the second gives rise to *The Geometry of Size;* and the third gives rise to *The Geometry of Position.*

**Reasons for Early Introduction of Informal Geometry.** Various reasons have been given for the early introduction of informal geometry into the curriculum.

1. *Historical.* It is generally agreed that *mathematics is the most ancient of all the sciences,* that it originated naturally through the necessary processes of *counting* which gave rise to arithmetic and *measuring* which gave us geometry. However, demonstrative geometry is no longer taught primarily for the facts of mensuration.

Mr. Betz, in particular, has made a very thorough study of the possibilities of a historical approach in the field of intuitive geometry. After much research in our large museums, supplemented by interviews with leading anthropologists, he has made some of his findings available.[3] His main conclusion is that the conscious evolution of form was aided materially by the development of the manual arts among primitive people, and that these manual activities, in their historic setting, constitute one of the most natural, profitable, and enjoyable means of introducing the pupil to a study of geometric forms. No chronological sequence can be established for the order in which geometric forms were first discovered or used, but *there is no doubt about the general soundness of the thesis that manual activities have always been a principal source of geometric training.*

Among the life situations from which mathematics originated, the ones most often listed as important are those which have to do with *food, clothing,* and *shelter.* For example, Mr. Betz tries to explain the origin of some of our most common geometric forms by relating them to the problem of shelter. Starting with the caveman who built no shelter, going through the stage where man built one form of shelter or another to the stone age where the houses resembled ours, we find some very interesting things.

The Indians, for instance, built cone-shaped tepees presumably because they were familiar with the evergreen tree and had often found shelter under its spreading branches. Moreover, the ancients doubtless lived in trees at first and then came down when weapons and fire were discovered. It is known that to this day primitive people build their shelters first out of the branches of the trees and then out of poles covered with skins or hides; that from the outline of the branches of the tree on the ground they discovered or first came to recognize the "circle" as a geometric form.

The construction of circles became necessary in building both tepees and wigwams (dome-like shelters). The first step taken was the selection of the center, and at that point a stake was driven into the ground. To this stake was attached a strip of skin or an improvised rope (vine), which served as a radius. This rope was "carried around" the center until a circle resulted. This is the literal origin of the term "circumference."

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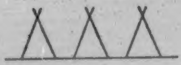
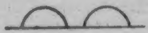
[3] See, especially, the following: 1) Betz, Wm., *Geometry for Junior High Schools,* published by J. M. Egloff, Rochester, N. Y.; 2) Betz–Miller–Miller, *Work Book in Intuitive Geometry,* published by The Harter Publishing Company, Cleveland, Ohio; 3) *Illustrated Talks by Pupils of the Lincoln School,* published by the Lincoln School of Teachers College, New York, N. Y.; 4) Betz, Wm., "Intuition and Logic in Geometry," *The Mathematics Teacher,* 2:3-31.

If the Indians tried to illustrate their work at all they doubtless used picture writing like that illustrated here to represent their tepees. This would suggest the "triangle" to any progressive savage.



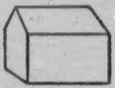
Next in order came dome-like shelters—hemispheres, so to speak, which might easily have been suggested by the shape of caves or of the sky overhead. These dome-like shelters were doubtless built by gathering numerous pliable reeds of suitable lengths and fastening the two ends in the ground to form what we know to-day as the "wigwam."

If a savage represented his wigwam by picture writing, he no doubt drew something which looked like this figure, and which of course suggested the semicircle to the aforesaid progressive savage.



It has been further suggested that the savages doubtless made "sweat-lodges" out of their wigwams by cutting a hole in the center into which they placed hot stones and water by means of which steam for the bath was generated. Then when some tall savage become too hot or desired to stand upright, someone got the idea of enlarging the place either by lifting the top, as one might unconsciously do, or by digging down into the ground. In either case a cylindrical base would result.

It is also known that primitive people often found it convenient to build an ordinary "lean-to" to break the more biting winds. Such lean-tos may still be seen in certain parts of our own country. When the weather became worse, a double lean-to no doubt was provided for the comfort of the family. Here again a tall savage who might have gone beneath the lean-to and then suddenly risen may have suggested the need for a raised lean-to. This would result in a shelter like that pictured here, with a rectangular solid as a base. This form of shelter has become the most common one among all civilized people. Hence the rectangle has become the most common of all the plane figures.



All these explanations are of interest to any normal child and from an imaginative standpoint have value in the course.

The historical approach, interesting as it may be, is not the only way to introduce children to the study of geometry. I have tried the recreational approach through the study of such a topic as "Other Spaces than Ours"[4] and the children take tremendous interest in the work besides learning many important concepts and skills.

2. *Pedagogical.* There are two reasons why geometry is so difficult for many pupils: first, the high school population to-day is so different from that of a generation ago. The pupils of to-day, as Thorndike[5] has pointed out, differ from those of a generation ago not only in native ability, but also in experiences and interests. This presents the first important problem. Second, in the tenth grade the pupil is plunged headlong into the study of formal geometry without any previous preparation in or experience with informal geometry as a background.

The next problem is, therefore, to consider the importance of beginning the study of geometry earlier and spreading it over a longer period of time. If we do this generally in this country many of the problems to be discussed later in this Yearbook will be more easily solved.

Professor Nunn has written some interesting comments upon what he considers fundamental stages in the student's development. He says:

> I assume as common ground that the school course in geometry should show two main divisions: (1) a heuristic stage in which the chief purpose is to order and clarify the spatial experiences which the pupil has gained from his everyday intercourse with the physical world, to explore the more salient and interesting properties of figures, and to illustrate the useful applications of geometry, as in surveying and "Mongean" geometry; (2) a stage in which the chief purpose is to organize into some kind of logical system the knowledge gained in the earlier stage and to develop it further. In the first stage obvious truths (such as the transversal properties of parallel lines) are freely taken for granted, and deduction is employed mainly to derive from them important and striking truths (such as the constancy of the angle-sum of a triangle) which are not forced upon us by observation. The second stage is marked by an attempt, more or less thoroughgoing and "rigorous," to explore the connexions between geometrical truths and to exhibit them as the logical consequences of a few simple principles.[6]

3. *Practical.* If we wish to make any kind of object, it must first be conceived in the mind. It must then be carefully planned as to size and the relation of its parts, and finally the parts must

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[4] See Smith, David Eugene, and Reeve William David: *The Teaching of Junior High School Mathematics*, pp. 308-314. Ginn and Company, 1927.

[5] Thorndike, E. L., *The Psychology of Algebra*, p. 2. Macmillan Co.

[6] Nunn, T. P., "The Sequence of School Theorems in Geometry." *The Mathematics Teacher*, October, 1925.

be made and carefully placed in their proper position. A tailor, for example, must think of shape, size (pattern), and position in his work just as we do in similar situations throughout life. Of course we can readily see that all such ideas originated in nature. One of my seventh-grade children once wrote in his notebook, "I think Nature must have been the first teacher of geometry." And so she was.

On the practical side of geometry a great deal of the work will be *form study*. The student will come into contact with a great many of the important forms through his senses. Here is where the teacher is so important in guiding the student to get the forms correctly. Many students as traditionally taught are blind to the forms in the world about them.

4. *Cultural.* Failure of students to appreciate the forms in life about them leads to failure to appreciate much of the beauty in the world. In a recent sermon Dr. Harry Emerson Fosdick said, "Beauty is a matter of symmetry." He might have added, because the fact is well known, "and symmetry is a matter of appreciation of form in geometry."

Vienna has been called the center of fine arts and well it may be, for there can be found some of the finest examples of form study in the world.

If whatever is made must first be seen in the mind, then Mr. Betz is right when he says that "all beautiful things are the result of *plastic thinking*." We teachers of mathematics need to train our students to appreciate beauty in form and design. Here is the place for "creative education"—for "creative thinking." Here is the place to feast the imagination, to let it *run riot*, so to speak; for, as Coleridge said:

> Whilst reason is feasting luxuriously in its proper paradise,
> Imagination is perishing on a dreary desert.

The trouble with so much of our teaching of geometry is that we "feast the reason and starve the imagination."

**Experience of Other Countries.** The experience of some of the European countries ought to convince us of the value of an informal geometry course in the seventh grade. For many years the mathematics teachers in Germany and England have given a great deal of attention to geometry in the earlier years of the child's school life. Even as early as 1876 Herbert Spencer wrote the following letter to D. Appleton and Company, who were then about to republish in the United States a book on "Inventional Geometry" written by his father, William George Spencer. This book had already appeared in England:

London, June 3, 1876

Messrs. D. Appleton & Co.: I am glad that you are about to republish, in the United States, my father's little work on "Inventional Geometry." Though it received but little notice when first issued here, recognition of its usefulness has been gradually spreading, and it has been adopted by some of the more rational science-teachers in schools. Several years ago I heard of its introduction at Rugby.

To its great efficiency, both as a means of producing interest in geometry and as a mental discipline, I can give personal testimony. I have seen it create in a class of boys so much enthusiasm that they looked forward to their geometry-lesson as a chief event in the week. And girls initiated in the system by my father have frequently begged of him for problems to solve during their holidays.

Though I did not myself pass through it—for I commenced mathematics with my uncle before this method had been elaborated by my father—yet I had experience of its effects in a higher division of geometry. When about fifteen, I was carried through the study of perspective entirely after this same method: my father giving me the successive problems in such order that I was enabled to solve every one of them, up to the most complex, without assistance.

Of course, the use of the method implies capacity in the teacher and real interest in the intellectual welfare of his pupils. But given the competent man, and he may produce in them a knowledge and an insight far beyond any that can be given by mechanical lesson-learning.

Very truly yours,

**HERBERT SPENCER.**

The author himself wrote the following introduction for the American edition:

### INTRODUCTION

When it is considered that by geometry the architect constructs our buildings, the civil engineer our railways; that by a higher kind of geometry, the surveyor makes a map of a country or of a kingdom: that a geometry still higher is the foundation of the noble science of the astronomer, who by it not only determines the diameter of the globe he lives upon, but as well the sizes of the sun, moon, and planets, and their distances from us and from each other; when it is considered also, that by this higher kind of geometry, with the assistance of a chart and a mariner's compass, the sailor navigates the ocean with success, and thus brings all nations into amicable intercourse—it will surely be allowed that its elements should be as accessible as possible.

Geometry may be divided into two parts—practical and theoretical: the practical bearing a similar relation to the theoretical that arithmetic does to algebra. And, just as arithmetic is made to precede algebra, should practical geometry be made to precede theoretical geometry.

Arithmetic is not undervalued because it is inferior to algebra, nor ought practical geometry to be despised because theoretical geometry is the nobler of the two.

However excellent arithmetic may be as an instrument for strengthening the intellectual powers, geometry is far more so; for as it is easier to see the relation of surface to surface and of line to line, than of one number to another, so it is easier to induce a habit of reasoning by means of geometry than it is by means of arithmetic. If taught judiciously, the collateral advantages of practical geometry are not inconsiderable. Besides introducing to our notice, in their proper order, many of the terms of the physical sciences, it offers the most favorable means of comprehending those terms and impressing them upon the memory. It educates the hand to dexterity and neatness, the eye to accuracy of perception, and the judgment to the appreciation of beautiful forms. These advantages alone claim for it a place in the education of all, not excepting that of women. Had practical geometry been taught as arithmetic is taught, its value would scarcely have required insisting on. But the didactic method hitherto used in teaching it does not exhibit its powers to advantage.

Any true geometrician who will teach practical geometry by definitions and questions thereon, will find that he can thus create a far greater interest in the science than he can by the usual course; and, on adhering to the plan, he will perceive that it brings into earlier activity that highly valuable but much-neglected power, the power to invent. It is this fact that has induced the author to choose as a suitable name for it, the "inventional method" of teaching practical geometry.

He has diligently watched its effects on both sexes, and his experience enables him to say that its tendency is to lead the pupil to rely on his own resources, to systematize his discoveries in order that he may use them, and to induce gradually such a degree of self-reliance as enables him to prosecute his subsequent studies with satisfaction: especially if they should happen to be such studies as Euclid's "Elements," the use of the globes, or perspective.

A word or two as to using the definitions and questions. Whether they relate to the measurement of solids, or surfaces, or of lines; whether they belong to common square measure, or to duodecimals; or whether they appertain to the canon of trigonometry; it is not the author's intention that the definitions should be learned by rote; but he recommends that the pupil should give an appropriate illustration of each as a proof that he understands it.

Again, instead of dictating to the pupil how to construct a geometrical figure—say a square—and letting him rest satisfied with being able to construct one from that direction, the author has so organized these questions that by doing justice to each in its turn, the pupil finds that, when he comes to it, he can construct a square without aid.

The greater part of the questions accompanying the definitions require for their answers geometrical figures and diagrams, accurately constructed by means of a pair of compasses, a scale of equal parts, and a protractor, while others require a verbal answer merely. In order to place the pupil as much as possible in the state in which Nature places him, some questions have been asked that involve an impossibility.

Whenever a departure from the scientific order of the questions occurs, such departure has been preferred for the sake of allowing time for the pupil to solve some difficult problem; inasmuch as it tends far more to the formation of a self-reliant character, that the pupil should be allowed time to solve such difficult problems, than that he should be either hurried or assisted.

The inventive power grows best in the sunshine of encouragement. Its first shoots are tender. Upbraiding a pupil with his want of skill, acts like a frost upon him, and materially checks his growth. It is partly on account of the dormant state in which the inventive power is found in most persons, and partly that very young beginners may not feel intimidated, that the introductory questions have been made so very simple.

**Early American Textbooks in Informal Geometry.** Several books written in this country many years ago show that the idea of an informal geometry course is not only not a new one, but has even been considered important by progressive teachers and thinkers for a long time. The preface of *First Lessons in Geometry*, written by Thomas Hill in 1854, which is now out of print, reads in part as follows:

> I have long been seeking a Geometry for beginners, suited to my taste, and to my convictions of what is a proper foundation for scientific education. Finding that Mr. Josiah Holbrook agreed most cordially with me in my estimate of this study, I had hoped that his treatise would satisfy me, but, although the best I had seen, it did not satisfy my needs. Meanwhile, my own children were in most urgent need of a textbook, and the sense of their want has driven me to take the time necessary for writing these pages. Two children, one of five, the other of seven and a half, were before my mind's eye all the time of my writing; and *it will be found that children of this age are quicker of comprehending first lessons in Geometry than those of fifteen.*[7]
>
> Many parts of this book will, however, be found adapted, not only to children, but to pupils of adult age. The truths are sublime. I have tried to present them in simple and attractive dress.
>
> I have addressed the child's imagination, rather than his reason, because I wished to teach him to conceive of forms. The child's powers of sensation are developed, before his powers of conception, and these before his reasoning powers. This is, therefore, the true order of education; and a powerful logical drill, like Colburn's admirable first lessons of Arithmetic, is sadly out of place in the hands of a child whose powers of observation and conception have, as yet, received no training whatever. I have, therefore, avoided reasoning, and simply given interesting geometrical facts, fitted, I hope, to arouse a child to the observation of phenomena, and to the perception of forms as real entities.

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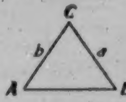
[7] The italics are Hill's.

I give this in some detail in order to show that the idea of an intuitive geometry course is not so recent in this country as some teachers seem to think. The large number of books[*] similar to Hill's, written since 1850, will be somewhat surprising to many teachers who are not friendly to the junior high school course in mathematics and who can think of a great many reasons for continuing to believe that what they already think is the last word in teaching geometry.

**Demonstrative Geometry.** As the human mind develops it does not demand merely probable facts; it seeks to prove that these facts are real. It is this attitude which has led the race to seek to demonstrate the truth of its statements concerning geometric figures.



If in intuitive geometry, for example, the teacher tells the pupil that in this figure $a = b$ and asks him to express an opinion on the relation between the size of angles $A$ and $B$, the pupil will naturally say that these angles are equal. It is not probable that any so-called logical proof that his inference is correct will noticeably increase his confidence in the validity of his judgment upon any proposition of this degree of simplicity.

When, however, the pupil meets the Pythagorean theorem, his intuition leads him to conclude its truth for only very special cases. His greater maturity of thought should now tend to lead him to desire to prove what is not so plainly evident as the theorem relating to the isosceles triangle. When that "urge" toward demonstration appears, he is ready for what is called *demonstrative geometry.*

To some minds this "urge" comes early; to others it comes late; to a few it comes not at all. Until it comes, however, the pupil can profit but little from the study of demonstrative geometry. He must accept on faith the simpler facts of geometry. When, if ever, he comes to appreciate the value of a demonstration, he will see the significance of the systematized scheme or chain of geometry and will then take an interest in proving most of the propositions which he once took for granted.

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[*] 1. Campbell, W. T., *Observational Geometry*, American Book Co., 1899.
2. Coates, J. V. H., *A First Book of Geometry*, Macmillan Co., 1912.
3. Failor, I. N., *Inventional Geometry*, The Century Co., 1904.
4. Fowler, C. W., *Inductive Geometry*, published by the author, 1905.
5. Hill, G. A., *A Geometry for Beginners*, Ginn and Co., 1897.
6. Hornbrook, A. R., *Concrete Geometry*, American Book Co., 1895.
7. MacDonald, J. W., *Geometry in the Secondary School*, Willard Small, 1889.
8. Row, T. S., *Geometric Exercises in Paper Folding*, Open Court Publishing Co., 1905.
9. Wright, D. S., *Exercises in Concrete Geometry*, D. C. Heath and Co., 1907.

For that matter, Euclid did not prove that if in the triangle $ABC$ we know that $a = b$, then $\angle A = \angle B$ *because anybody doubted it*, but because it was a part of his logical chain of geometry.

**Deduction and Proof.** It will be of interest here to note Carson's distinction between "deduction" and "proof." He says:

> Turning now to the more educational aspect of the subject, the first problem which confronts us is this: children, when they commence mathematics, have formed many intuitions concerning space and motion; are they to be adopted and used as postulates without question, to be tacitly ignored, or to be attacked? Hitherto teaching methods have tended to ignore or attack such intuitions; instances of their adoption are almost non-existent. This statement may cause surprise, but I propose to justify it by classifying methods which have been used under one or other of the two first heads, and I shall urge that complete adoption is the only method proper to a first course in mathematics.
>
> Consider first the treatment of formal geometry, either that of Euclid or of almost any of his modern rivals; in every case intuition is ignored to a greater or less extent. Euclid, of set purpose, pushes this policy to an extreme; but all his competitors have adopted it in some degree at least. Deductions of certain statements still persist, although they at once command acceptance when expressed in non-technical form. For example, it is still shown in elementary text-books that every chord of a circle perpendicular to a diameter is bisected by that diameter. Draw a circle on a wall, then draw the horizontal diameter, mark a point on it, and ask any one you please whether he will get to the circle more quickly by going straight up or straight down from this point. Is there any doubt as to the answer?[*] And are not those who deduce the proposition just quoted, from statements no more acceptable, ignoring the intuition which is exposed in the immediate answer to the question? All that we do in using such methods is to make a chary use of intuition in order to reduce the detailed reasoning of Euclid's scheme; our attitude is that statements which are accepted intuitively should nevertheless be deduced from others of the same class, unless the proofs are too involved for the juvenile mind. *We oscillate to and fro between the Scylla of acceptance and the Charybdis of proof, according as the one is more revolting to ourselves or the other to our pupils.*[**]
>
> At this point I wish to suggest that a distinction should be drawn between the terms "deduction" and "proof." There is no doubt that proof implies access of material conviction, while deduction implies a purely logical process in which premises and conclusion may be possible or impossible of accept-

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[*] There is often apparent doubt; but it will usually be found that this is due to an attempt to estimate the want of truth of the circle as drawn.

[**] The italics are mine.—Editor.

ance. A proof is thus a particular kind of deduction, wherein the premises are acceptable (intuitions, for example), and the conclusion is not acceptable until the proof carries conviction, in virtue of the premises on which it is based. For example, Euclid deduces the already acceptable statement that any two sides of a triangle are together greater than the third side from the premise (inter alia) that all right angles are equal to one another; but he proves that triangles on the same base and between the same parallels are equal in area, starting from acceptable premises concerning congruent and converging lines.

The distinction has didactic importance, because pupils can appreciate and obtain proofs *long before they can understand the value of deductions; and it has scientific importance, because the functions of proof and deduction are entirely different.*[u] Proofs are used in the erection of the superstructure of a science, deductions in an analysis of its foundations, undertaken in order to ascertain the number and nature of independent assumptions involved therein. If two intuitions or assumptions, *A* and *B*, have been adopted, and if we find that *B* can be deduced from *A*, and *A* from *B*, then only one assumption is involved, and we have so much the more faith in the bases of the science. Herein lies the value of deducing one accepted statement from another; the element of doubt involved in each acceptance is thereby reduced.

## II. PURPOSE OF STUDYING GEOMETRY

**Purpose in Teaching Demonstrative Geometry.** The first important question for any teacher of demonstrative geometry to settle is the purpose he has in mind. A great deal of our failure to agree on certain matters of curriculum construction is due to the fact that we do not agree on the valid aims in teaching the various topics. That is why senior high school teachers of geometry so often object to the teaching of informal geometry below the tenth grade. The common objection is that if any geometry is taught below the tenth grade the pupils who come to them will be handicapped in the work of the tenth grade. This is largely a misconception of the real situation and is due to lack of understanding of the purpose in teaching informal geometry in the seventh and eighth grades or a unit of demonstrative geometry in the ninth grade, or both.

We must continually repeat and emphasize the fact that the purpose of geometry is to make clear to the pupil *the meaning of demonstration, the meaning of mathematical precision, and the pleasure of discovering absolute truth.* If demonstrative geometry is not taught in order to enable the pupil to have the satisfaction of proving something, to train him in deductive thinking, to give him the power to prove his own statements, then it is not worth teaching at all.

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[u] The italics are mine.—Editor.

Someone may ask, If training in constructive thinking is the big objective, why not give a course in pure logic? The answer is that geometry furnishes appropriate figures to illustrate and apply the essential types of thinking, while pure logic does not.

**Who Can Profit by Studying Geometry?** Informal geometry represents about all the geometry that many pupils are capable of understanding. If all pupils had strong native ability, they could dispense with a considerable part of informal geometry and could proceed with only a slight introduction to the demonstrative stage. As it stands to-day, however, with almost all children going to secondary schools, informal geometry constitutes about all the training in geometry that a considerable body of our pupils will be able to absorb with profit.

On the other hand, the pupil with a fairly high intelligence quotient will derive more pleasure and interest from doing than from merely depending upon his intuition; he will obtain satisfaction in demonstrating that his intuition is correct. To do this he must learn to give reasons for every step in the thinking process; in other words, to demonstrate. This applies to all subjects. The natural sciences are not taught from a vocational point of view, but for purposes of general information and because they furnish the student an opportunity to collect and test empirical facts. Similarly, mathematics in general and demonstrative geometry in particular are not taught to make engineers out of those who study them; the former is taught for the general information it affords in one of the great branches of knowledge so useful in life; and the latter is taught because it gives exercise in deductive thinking where demonstration is independent of external appearances.

If we can develop some kind of prognostic test in demonstrative geometry that will tell us who can profit by such work, or if in some way we can select those who will be able to succeed in the study of the subject, any pupil whose mentality indicates a probable low degree of success should be excused from taking geometry. We must insist, however, that the responsibility of saying who can profit by a study of geometry and who cannot profit by it must rest on people qualified to give the proper advice. That is another reason why a trial course in demonstrative geometry in the ninth grade appeals to so many teachers when the purpose of teaching it is clearly understood. From such a course it should be possible to predict with some measure of accuracy whether a pupil ought to go on with the study of demonstrative geometry in the tenth grade.

**Geometry the Supreme Test.** For some the study of geometry is the real test of a real and abiding interest in mathematics. Professor Nunn told me that he got his first real thrill when he discovered for himself that the sum of the angles of a triangle is 180°. It is related that Newton showed little promise until he took up the study of geometry. The point is that we never can tell what a pupil will do if he is properly aroused.

**Omissions and Additions.** In line with the purpose of teaching geometry previously stated, we need to investigate carefully the conditions which work against obtaining the best results in the teaching of geometry, and to discover the possible means of remedying existing imperfections. These imperfections may be remedied by omitting some of the unnecessary and unimportant phases of the geometry work, by substituting better material, and by adding such new subjects or ideas as shall make the subject matter more desirable from the standpoint of both pupil and teacher. Moreover, we need to improve our methods of teaching.

We have all heard the complaint that students are unable to apply to other sciences the principles which they are supposed to have learned in mathematics. It is such complaints as this and the realization of their justness that have caused mathematicians to turn their minds to the improvement of the subject matter in the study of secondary mathematics. Dr. Osgood of Harvard once said, "A student's ability to prove a proposition is no assurance that he knows it. The test as to whether he knows it is whether he can use it."

In no event will all authorities agree upon the best plan for betterment. There is a tendency to go from one extreme to the other, but it ought to be possible to study the situation intelligently and then adopt whatever plans are considered best suited to particular needs.

It has been suggested that all propositions not bearing on subsequent work in the same course should be omitted. Some teachers disapprove of this plan because many such propositions give pleasure and make the work interesting to the student. Certainly we want to make the work as interesting as possible, but should we keep such material in the course, even though interesting in the classroom, if something better, bearing on subsequent material, is to be had?

Others say that much time is wasted in work on original exercises which by many teachers are made a fetich, that "a proposition whose solution is given by the teacher or by one or two pupils out of a class of forty, for the amazement of interested dullness, or the vacant stare of apathetic indolence of the class, to be copied mechanically as an exercise in penmanship and drawing, is no more an original than any of the propositions in the regular text." If, as has been said, the object of preparatory instruction is twofold—(1) to put the pupil in possession of certain facts, and (2) to develop in him mathematical power—it is equally true that the routine method of handling originals does very little toward accomplishing the first object and practically nothing toward attaining the second.

The geometry report of the Central Assocation of Science and Mathematics Teachers for 1908 is full of suggestions as to omissions and additions. It says, "The proposals of the committee are meant to be evolutionary, not revolutionary in tendency; and each suggestion is intended as one which could be adopted without departing far from current texts and methods. The committee expects no sudden, drastic reforms in the teaching of geometry, but hopes that those who may have been accustomed to look upon current texts as ideal and their logic as unassailable will tolerate criticism and concede the desirability of reforms. In any event, it is not advised that many changes be introduced hastily, but that, so far as possible, the teacher become familiar with each change before putting it into practice in teaching."

**Criticisms and Suggestions.** It would be difficult to list all the criticisms and suggestions that have been made for improving the situation in geometry. The following are a few of those most frequently offered.

1. All geometry has cultural value. Omit "culture for culture's sake" and teach a more practical geometry.
2. The question of rigor is overrated and overdone.
3. There is too much hair-splitting of definitions, and too much time wasted defining terms.
4. Too many theorems that need no proof are proved. Intuition or experience should be substituted in such cases.

5. More material is crowded into a single year than a child fourteen or fifteen years of age can absorb.

6. The notation in many textbooks is too elaborate, and besides it obscures the thread of logic. For example, a line can be represented by a single letter and so can an angle.

7. Textbooks in geometry do not discriminate carefully or sufficiently between the essentials and the nonessentials. They deluge the pupil "with a great number of stock propositions a large proportion of which are unimportant theorems."

8. Of all the suggested omissions in geometry the subject of limits has been given as much attention as any other part of geometry. Nearly every mathematics teacher who has suggested any omissions in geometry at all, so far as I know, has suggested at least less teaching of the theory of limits. Fortunately, most teachers of geometry now omit all reference to this subject, although it has taken a long time to bring about this change. In fact, some of the newer geometries still give it space in a supplement.

Professor Alan Sanders, in the opening address to the Association of Ohio Teachers of Mathematics and Science at Columbus, December 29, 1904, discussed at length the question of limits and showed why the subject was introduced into our American texts. He also took up specific examples of the use of the theory of limits and showed the inconsistency in their application to particular proofs. He closed his argument with the following statement:

> The fact that 90 per cent of the geometries used in this country give no proof for the propositions quoted relating to cylinders, cones, and spheres is a matter worthy of the serious consideration of the teachers of geometry.

Professor Lennes, in commenting on what Mr. Sanders said at that time, interpreted the latter by saying that what Mr. Sanders meant to say was that the proofs *attempted* do not as a matter of fact prove anything. And he then adds, with an exclamation to the shades of Euclid:

> All the teachers of geometry in this country combined can no more prove these theorems than they can raise a two-year-old calf in a day.
>
> It is a "sine qua non" that shoddy proof must go. Clearness and honesty of thought is the ideal of mathematics if of any field of intellectual activity. If it should not be found possible to formulate a solid treatment of limits adequate to meet the needs of elementary geometry and at the same time sufficiently simple to be understood by the pupil, then either the topics

where limits are now used must be omitted or a comprehensible treatment must be devised without the use of limits. In case the latter alternative were chosen, one might consider only those segments, angles, etc., which are commensurable. Of course, it would have to be stated explicitly that such treatment is not complete but that we assume the theorems without proofs in case the segments, etc., are incommensurable. The length of a circle might be considered as the perimeter of a regular polygon of, say, one million sides, etc. Our results would then be accurate within the limits of observation. At every step we should have to be perfectly clear as to what we are doing. The treatment could easily be managed so as to be entirely logical throughout.

## III. EXTENT OF THE COURSE

**Conditions Controlling the Selection of Content.** The pressure upon the curriculum, the new subjects that are clamoring for a place in the sun, and the demands that we break with tradition—all these problems have caused educators to question the extent to which geometry should be carried. The wisdom of teaching solid geometry in particular has been seriously questioned. In fact, solid geometry as a separate half-year course is rapidly becoming *passé* in our schools. It is not even required for entrance in some of our engineering schools and colleges, as, for example, the following statement shows:

> Solid Geometry.—Those who take this course have already had one year of Plane Geometry to provide a foundation in geometric methods and processes. In Solid Geometry much stress should be laid on the development of space perception. Pupils should be encouraged to make models, and then to draw them. The mensuration formulas relating to polyhedra, cylinders, cones, and spheres should receive special attention.
>
> This subject is no longer required for admission to any college of the University of Nebraska, or of a considerable number of other large universities. It is therefore being dropped from the curricula of many high schools. Substitution of a fourth semester of Algebra will be accepted by the University from those students who wish to offer six points entrance credits in Mathematics. Such students may then enter with three points in each subject, including Solid Geometry, as heretofore.
>
> Schools which do not offer a course in Solid Geometry would do well to see that the standard mensuration formulas from this subject are used sufficiently both in Algebra and in Plane Geometry to make the pupils quite familiar with them. They should include the formulas for the surface and volume of parallelopiped, cylinder, cone, and sphere. Proofs may be omitted.[13]

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[13] *University Extension News*, Vol. 8, No. 13, July 1928. University Extension Division, University of Nebraska.

**A Combination Course in Plane and Solid Geometry.** I am not taking the position that solid geometry should go or that it should stay, but I do wish to point out that if it is to stay it will probably have to be combined with plane geometry and be taught in the tenth year. In answer to this suggestion some teachers will complain that they cannot teach plane geometry in one year, much less solid geometry. Such an attitude presupposes a definite amount of "ground" to be covered and this is a myth, except in the case of those who are bound by extramural examinations. We cannot teach all of geometry or all of any other subject in a lifetime. Moreover, we do not teach geometry for the purpose of teaching any given list of propositions, but to develop the ability to demonstrate. The fact is that many pupils can learn all that is worth learning in plane and solid geometry, and many others will learn little if any. The rule of common sense—not, as is so often the case, the rule of prejudice—should determine the extent of the course.

Some teachers say that they wish to preserve a distinct place for solid geometry in the schools because it is the subject which teaches spatial relationships. However, trigonometry will do that as well or better and it has other advantages not credited to solid geometry.

If solid geometry is to be preserved, it can be done in three ways:

1. It can be taught as a separate unit in the tenth grade much as it is done to-day, except that it must be more concentrated.
2. It can be fused with plane geometry at the places where such fusion seems most desirable and possible.
3. It can be taught intuitively.

The advantages and disadvantages of each plan should be discussed and some intelligent conclusion reached.

**European Practice.** In the European schools the number of propositions studied is generally smaller than has until recently been the case with us, and the teaching of the subject is extended over a longer period of time.[13] Our practice must be more uniform than the practice in European schools because our people move about so much, and this makes it harder for us to arrange our courses and the sequence of work to suit local needs. Besides, European schools take several years to do what we crowd into one year with disastrous results. The rise of the junior high school idea, however, has enabled us to take advantage of some of the best ideas in European practice.

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[13] All teachers of mathematics in this country should be familiar with the report of the British Association entitled *The Teaching of Geometry in Schools. A Report prepared for the Mathematical Association.* G. Bell and Sons, London.

Along with the development of the junior high school course in informal geometry we have also reduced the traditional number of plane geometry propositions[14] which we expect a student to master and have increased considerably the number of original exercises. Not only that, but these exercises are more carefully chosen. As a result, they are simpler, more interesting, and on the whole better adapted to our purpose of having the majority of our students acquire the method of demonstration. We should not limit the work to only those who possess rare mathematical ability and who expect to be experts.

**Algebra in Geometric Proofs.** There has been a great deal of discussion concerning the advisability of using algebra in geometric proofs. The purists are in favor of leaving algebra out altogether, in spite of the fact that the modern view of mathematics permits geometry and analysis to complement each other. At the other extreme are those who try continually to force algebra into a geometric proof whether or not it properly belongs there.

A more satisfactory view would seem to be to use algebra in a geometric demonstration when the failure to do so would make the proof unusually difficult. Likewise, in such propositions as the Pythagorean theorem it is not only easier but better to generalize the theorem by including the other two metrical cases about the square on the side opposite an acute angle and the square on the side opposite an obtuse angle, and to use both algebra and trigonometry in proving it. Such a procedure will save time that can be devoted to the salient features of solid geometry.

## IV. FOUNDATIONS OF GEOMETRY

**Fundamental Principles and Definitions.** The traditional plan of proving propositions is to refer back to statements already proved. However, there is no proposition before the first one by means of which the first one is proved; so we decide to set up certain statements that we are willing to accept without proof. These conventional statements are called *postulates*. Besides postulates we use certain general assumptions, common to all mathematics, called *axioms*. It is now generally understood that the number of assumptions should be small, but not too small.[15]

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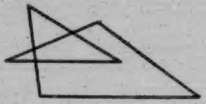
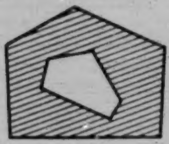
[14] See the College Entrance Examination Board's Syllabi in Algebra and Geometry. Documents Numbers 107 and 108.

**Nature and Purpose of Definitions.** Whereas the modern tendency is to omit all memorized definitions in arithmetic, and for the most part in algebra, the opposite is true in geometry. Here the proof of a proposition may be based upon certain definitions as well as upon preceding proofs or assumptions (postulates or axioms). As examples of terms which are used in proofs and which should therefore be memorized, we have *right angle, perpendicular, bisector, perpendicular bisector,* and so on. Here again, the textbook or the teacher should guide the pupil in deciding which definitions need to be memorized.

It is not possible to define all terms adequately; some of them must simply go undefined. Such terms as *point, line, plane,* and *angle* are good examples of things that might better go undefined. The failure to memorize definitions for these terms does not, however, excuse the pupil for using them improperly. He is required to use the term *electricity* although he cannot define it.

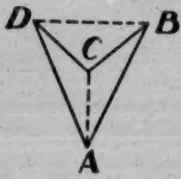
**How Precise Must a Definition Be?** All teachers of geometry who know anything about the situation at all, know that precision of definition is very difficult. Let the teacher who feels that absolute accuracy is necessary first define a polygon and then decide whether these figures come within his definition. If this causes him no worry, let him figure out the smallest number of vertices that a polygon may have, and then ask himself whether a regular *digon* (two-angled polygon) is possible, and if not, why not.





Let him further inquire whether the diagonals of a quadrilateral lie inside or outside the figure, and consider such a simple case as the one here shown, in which the diagonals are $AC$ and $BD$.

Finally, let him consider whether a quadrilateral is formed by intersecting lines, and how many diagonals are possible, and then see if his statements meet all the conditions suggested by the figure here shown.



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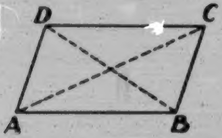
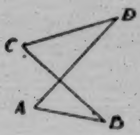
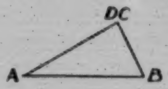
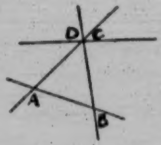
[15] For a list of such assumptions the reader is referred to any good modern text in geometry.

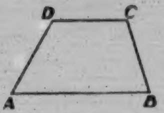
Such questions should not be introduced in the early stages of geometry. At that time a pupil may properly think of a quadrilateral as convex, as in the accompanying figure. This figure, as he properly conceives it, has only two diagonals, shown by the dotted lines $AC$ and $BD$. The teacher, however, should realize that in general a quadrilateral is formed by four straight lines lying in a plane. If we adopt the modern phraseology we say that these lines intersect in points at a finite distance or at an infinite distance, and that in any case we have three diagonals, even though some may be infinitely far away. For example, there are simple cases that may be profitably considered by the teacher and perhaps referred to later in the pupil's course similar to these illustrations:











Such considerations may suggest more forcefully to the teacher the undesirability of committing to memory definitions that are not actually used in proofs, since we are constantly extending our ideas of even very common terms, and any claim for absolute precision of definition is sure to be a hindrance to progress and will probably be withdrawn as we proceed in our work.

**Distinction between Theorems and Problems.** The *theorems* of geometry are concerned with proving geometric statements; the *problems* are concerned with the construction of geometric figures in a plane, the only instruments allowed being an unmarked straight-edge and a pair of compasses. In solid geometry we assume that the necessary figures can be constructed, and so we do not attempt to show how this can be done. The term *proposition* is commonly used to cover both *theorem* and *problem*.

In early days, upwards of two thousand years ago, the writers on plane geometry did not attempt to prove any theorem until they had shown that the figure could be constructed. For this purpose they placed some of their problems of plane geometry first, and introduced others as needed. At present we generally assume that all figures in plane geometry can be constructed, as we assume for solids, but we prove this later in a number of important cases.

We might go even farther, for it would be possible and logical to assume all constructions in plane geometry, just as we assume them in more advanced work. We do this in some cases, as it is; for example, we would not hesitate to ask a student to tell the number of right angles in the sum of the interior angles of a regular seven-sided polygon, although it is impossible to construct such a figure with the limitation imposed upon plane geometry; namely, of using only the compasses and the straight-edge.

Because of this modern view of the case, we generally place the problems at the end of any particular book or chapter although they might, as with ancient writers, be scattered among the theorems.

**Number and Importance of Theorems.** The number of possible theorems in plane geometry, and similarly in solid geometry, is apparently unlimited. New propositions are continually being discovered, most of them being simple deductions from propositions already known. The latter are usually theorems upon which a considerable number of others depend; that is, they may properly be designated as basic propositions.

Among the basic propositions the following are particularly important:

1. The congruence theorems.
2. The equality of alternate angles in the case of parallels.
3. The sum of the angles of a triangle.
4. The theorems relating to similar figures.
5. The Pythagorean theorem.
6. Measurement of an angle formed by two chords.

So important are these few propositions that, if we had no others with which to work, we could, with these alone, prove a large proportion of the original exercises of geometry. Indeed, it would not be bad practice to postulate all but the third and spend the time finding out how many exercises could be mastered with these as a foundation. A pupil would be fully as well off so far as mere training in logic is concerned, although he would not have such helpful mathematical equipment as he would have if he followed the usual plan.

Since the number of basic propositions in plane geometry is often assumed to be about one hundred, or hardly more than three for every school week, have we time under this plan for the important part of geometry—the exercises; indeed, have we time for the alternative plan of giving a fair idea of solid geometry in this same year? It is a mistaken idea that the best results require the repetition of the proof of every proposition in any particular textbook. A teacher who is able to arouse the interest of his pupils in independent work with the exercises, or in finding other exercises that are new to them, may safely discuss in class the proofs of the less important theorems, with a brief citing of the reasons involved in each step.

**Model Proofs.** Given this number of basic propositions, the question arises as to how they should be presented. Should we give the proofs in full? Should we give them in full at first and gradually leave more and more gaps for the pupils to fill in order to make the proofs complete? Should we dictate the propositions and have the pupils work out the proofs? Should we follow a syllabus instead of dictating, still leaving the proofs to be worked out? Should we employ intuition, pretend to discover the propositions, and then invent our own proofs, perhaps working them out by having the entire class take part, we ourselves guiding them in the right track? Should we give suggested proofs, the pupils following out the suggestions, ourselves pretending to encourage an originality which the suggestions render impossible? Or should we make some other combination or experiment, knowing very well that the same thing has doubtless been attempted many hundreds or thousands of times before? Given a teacher with enthusiasm and personal magnetism, any one of these plans will yield fairly good results.

The plans are not equally good, however, and world experience has generally favored the use of a textbook that gives the proofs of the early propositions in full, gradually reducing the degree of completeness and leaving the pupil more and more upon his own responsibility in completing the demonstration.

The purpose in giving a complete proof at first, with the reasons stated both by section number and in full form, is that the pupil may have a model before him. The reason for giving substantially complete proofs thereafter is that an approximate model may be before him even after he has come to rely more fully upon himself. It should never be assumed that proofs are given only to be memorized; they are given in order that a student should have, every day or two, a model for his treatment of the important exercises, these constituting the field in which his originality, his insight into geometry, and his ability to think logically are to be shown.

Whether the pupil writes his reasons under each statement of the proof or at the right of the statement is a matter of little moment. In the printed page a larger type and a more striking arrangement can be used if the reasons follow the steps, but in written work on a large page it is quite allowable to place reasons at the right, and many teachers prefer this arrangement because they find that they can the more readily mark the papers when they are written in this way.

In any case, the model in the textbook will serve to keep before the student the necessity for succinct and logical expression.

**Euclid's Sequence Versus the Modern Sequence.** For about 2,000 years Euclid's sequence was the order that was universally followed. To-day we have a simpler and more usable sequence, not so rigidly scientific as Euclid intended it for university students, but within the reach of high school pupils and better adapted to their needs.

Euclid was little concerned with the classification of propositions. He arranged his propositions in an order that seemed to him to begin with the easiest proposition. He then built the superstructure so as to construct his figures before using them. We attempt to classify our propositions, but we do not attempt to construct our figures before using them. From the standpoint of strict logic Euclid's plan is better; for teaching purposes ours is superior. A good plan for teachers to follow is to choose a carefully written modern textbook in geometry, follow the sequence given there as carefully as possible, using all the ingenuity they can, emphasize the work on original exercises, and cultivate the originality and imagination of the pupils as much as possible.

## V. METHODS OF ATTACKING ORIGINAL EXERCISES

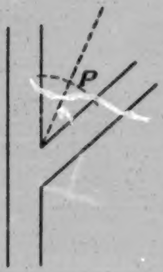
**Four of the Methods.** No single method of attack can be applied to every exercise. This is fortunate because otherwise our teaching would be more formal and wooden than it is. However, it is worth something to point out to a pupil some of the more definite methods of attack so that the "trial-and-error" method may not be overworked to the loss of all concerned. The most commonly used methods will be discussed.

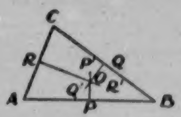
1. *The Synthetic Method.* Professor David Eugene Smith says of this method:

> The pupil usually wanders about more or less until he strikes the right line, and then he follows this to the conclusion. He should not be blamed for doing this, for he is pursuing the method that the world followed in the earliest times, and one that has always been common and always will be. This is the synthetic method, the building up of the proof from propositions previously proved. If the proposition is a theorem, it is usually not difficult to recall propositions that may lead to the demonstration, and to select the ones that are really needed. If it is a problem it is usually easy to look ahead in the proposed solution, to see what is necessary for its accomplishment and to select the preceding propositions accordingly.[10]

2. *The Analytic Method.* The analytic method of attacking original exercises in geometry is generally recognized as one of the most powerful methods which the pupil can learn. For this reason Professor Schlauch's chapter on the analytic method will be of especial interest and no further discussion of it will be given here.

3. *The Method of Loci.* This method of attack in geometry applies chiefly to problems where some point is to be determined. This is the method of the intersection of loci. Thus, to locate an electric light at a point eighteen feet from the point of intersection of two streets and equidistant from them, evidently one locus is a circle with a radius eighteen feet and the center at the vertex of the angle made by the streets, and the other locus is the bisector of the angle. The method is also occasionally applicable to theorems. For example, suppose that we have to prove that the three perpendicular bisectors of the sides of a triangle pass through the same point; that is, that they are concurrent. Here the locus of points equidistant from *A* and *B* is *PP'*, and the locus of points equidistant from *B* and *C* is *QQ'*. These can easily be shown to intersect, as at *O*. Then *O*, being equidistant from *A*, *B*, and *C*, is also on the perpendicular bisector of *AC*. Therefore those bisectors are concurrent in *O*.[11]





4. *The Indirect Method.* This method is occasionally used as a last resort. It is severely condemned by many, but it has its supporters as well. In order to help teachers who find a treatment of this method difficult or unsatisfactory Professor Upton has prepared a chapter on the indirect method.

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[10] Smith, David Eugene, *The Teaching of Geometry*, p. 161. Ginn and Co., 1911.
[11] *Ibid.*, p. 163.

**General Directions for Proving Propositions.** Aside from the conventional methods of attack on exercises already referred to there are a few general directions for proving propositions that may be given to the pupil.

1. *Read the proposition carefully.* Many pupils fail to prove propositions because they do not read them carefully, determining clearly what is given and what is to be proved, and because they do not sketch free-hand a figure representing the conditions.

2. *Draw a general figure.* Failures in proving original exercises are often due to the fact that a pupil takes a special case of a figure and having proved the exercise assumes that his proof is general.

3. *Draw the proper figure.* The careful construction of the figure under consideration will often suggest the relations which lead ultimately to the proof.

4. *Decide definitely what is given and what is to be proved.* The given part can perhaps best be stated by using letters or symbols relating to the figure, and similarly for what is to be proved.

5. *Think out a careful plan for the proof.* Here it is worth while for the pupil to know the various ways of proving lines equal, angles equal, lines parallel, triangles congruent, and the like, so that he can be more intelligent in his selection of the proper plan to follow.

## VI. The Conduct of the Recitation

**Misuse of the Blackboard.** It is doubtful whether there is any way in which we have wasted more time or developed worse habits among our pupils than in the traditional misuse of the blackboard. The practice of sending an entire class to the blackboard to draw figures and write out proofs that they have probably memorized is wasteful of time and encourages bad habits. If someone held a stop-watch on us while we passed to and from the blackboard the amount of time lost throughout the country in one day would astonish us.

It is customary in some schools for the teacher to send a pupil to the board occasionally to draw a figure and give a demonstration, but it is usually very unsatisfactory because of the loss of time and inattention thus developed. If it is necessary to have complicated figures drawn, it is much more economical of time and it is better teaching to use large pieces of cardboard upon which neat and accurate drawings have been made.

**Discussing Proofs.** The proofs of the propositions as far as they are given in the textbooks should be models or they should not be proved at all. In fact, we could easily omit the proofs of many conventional propositions and the pupil would gain in every respect by such omissions. Why should we give the proofs for each of the family of parallelogram propositions which usually begins with the one which reads: "The diagonal of a parallelogram divides the parallelogram into two congruent triangles"? This and all such simple propositions should be treated as original exercises, provided, of course, the number of model proofs in the text is sufficient to give the pupil a good notion of a model proof.

**General Conduct of the Recitation.** An entire book might be written on the conduct of the recitation in geometry. We might discuss the various types of lessons to be recognized and taught, how to take care of the routine factors of the recitation, such as calling the roll, ventilating, and so on, how the time of the class should be employed and divided, when the assignment of home work should be made, how to ask artistic questions, and many other such problems. However, most people who are qualified to teach geometry at all should have had courses in method or in professionalized subject matter before they are permitted to teach. In such courses all these details should be discussed at greater length than time here permits. Besides, teaching is an art and there are always those who contend that there is no best way of teaching anything. Whether or not this is true I do not know. I do know that the teaching of geometry in this country needs to be and can be improved. It is with the hope that the general level of the teaching of geometry may be improved that this yearbook has been prepared.

# WHAT SHALL WE TEACH IN GEOMETRY?

By W. R. LONGLEY
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**Revolution in Mathematics.** Geometry is one of the oldest of educational disciplines. More than any other it has retained its essential character for centuries. Why, then, should this most stable subject of our curriculum be questioned at the present time? Why should we ask: What shall we teach in geometry? Greatly accelerated changes in all conditions of the civilized world have put teachers and advocates of mathematics in general on the defensive. This is particularly true in the more elementary work. For various reasons there is pressure from many sources to cut down the time devoted to instruction and the material included in the courses.

The attacks begin on arithmetic. I do not know how widespread these have been, but I do know that some educators maintain that much useless material is included in the courses in arithmetic in our elementary schools. As a specific illustration it is argued that the subject matter of fractions is much inflated and that no one in ordinary life uses a fraction whose denominator is anything but 2, 3, 5, or some simple multiple thereof, like 6, 10, or 12. Hence we should omit all work in fractions except simple addition and multiplication of fractions with the denominators just mentioned.

A little higher in the scale we are all familiar with the attacks made on the amount of time devoted to mathematics in the secondary schools. Not so many years ago one of the important associations of educators in New England formally adopted a resolution urging the restriction of work in algebra to one year. Other mathematical subjects have likewise been under fire. Thus far the line has held fairly well but some losses have been sustained by the mathematical army, the most serious, I think, being that on the solid geometry front.

**The Place of Mathematics.** Now the changes that have already been made and those which are at present advocated by professional educators are all in the line of progress. It should not be our attitude to combat progress simply because we are born conservatives or because we wish to hold on to our present jobs. We may well ask: Is there any justification for continuing to teach the present amount of mathematics to the large number of unwilling pupils who are our victims? A partial answer to the question is to be found by examining tendencies in our institutions of higher learning. That all engineering students need a considerable amount of mathematics goes without question. That not all college students need mathematics is attested by the fact that few colleges of liberal arts require mathematics of all students. Yet the amount of mathematics required of a growing group is increasing rapidly. Mathematics has always been the basic tool of the physicist and the astronomer, and modern theories involve more advanced mathematics than the earlier ones. The equipment of the present-day chemist involves more mathematical training. Students in the chemistry course at Yale are now required to have the two years of college mathematics given to engineers, and at the present moment we are considering a request from the department of chemistry to give a special third year course for their students. Everyone is familiar with the great increase in applications of mathematical statistics in the fields of economics, business, education, and the natural sciences. Perhaps not so many realize, however, the growing demand for mathematically trained men and women in medical research. A short time ago a portion of the staff of the Medical School of Johns Hopkins formed a class and requested a member of the department of mathematics to give them a course in the calculus. A similar need for more mathematical knowledge has been experienced by certain members of the faculty of the Yale Medical School. One of the physicians has felt that Yale College should have required him to study the calculus. He knew when he entered college that he wanted to be a physician. He was required to take physics, chemistry, zoölogy, but not mathematics. How should he know as an undergraduate that in a few short years he would find it so desirable to have a working knowledge of the calculus?

The total number of college students has increased so enormously that the number taking no mathematics may readily give the impression that other subjects are growing more rapidly in importance. But there can be no doubt that the number who need mathematics is also increasing rapidly, and the amount of mathematics required in each field of application has multiplied many fold. This situation should be sufficient justification for holding our mathematical trenches against every assault.

**Value of Geometry.** We should then be prepared not only to defend ourselves from any attacks that may be made on our present teaching of geometry, but also to make a vigorous counter-assault to retrieve what has been lost in solid geometry. First of all, we should take account of stock. Why should geometry be taught by us and why should it be studied by the pupils? The main reasons may be summarized under four heads.

1. *Logical exposition.* In this phase of the work nothing else approaches geometry. Nowhere else does the pupil marshal his facts in such strict order and present them with such precision of statement. For ages this has been considered the chief value of the study. Perhaps it will continue to be so, but we must not be unmindful of the fact that there are some heretics, that some psychologists assert that there is little, if any, transfer of training, and that a study of the water-tight arguments of geometry helps a lawyer very little in making a convincing argument in a courtroom. We can not rely too much on training in argument, from hypothesis to conclusion, as justification for retaining geometry as a required subject for all students.

2. *Geometric facts and relations.* The utility of this phase of geometry is too obvious to require comment. Everyone knows that to a large extent geometry has been indispensable to civilized man since the earliest days of which we have any record. Some of the oldest hieroglyphics that have been deciphered refer to records of land measurement in simple geometric forms. Some of the most complicated geometric forms and relations are involved in the latest triumphs of engineering construction. I should like, however, to call attention to the importance of a knowledge of geometric facts as a part of a liberal education. Most of the great horde of students going to college now are seeking a liberal education. They want to be able to appreciate what they read and to talk intelligently of art, literature, scientific progress, and current events in general. They would be greatly mortified if they failed to understand such a reference as "exacting his pound of flesh." Yet a similar ignorance of geometry is passed off lightly. Certain facts, for example, that a straight line is the shortest distance between two points, appear to be obvious. Others do not. For instance,

do all persons possessing what is called a liberal education know that on the surface of a sphere the shortest distance between two points is along the arc of a great circle? They do not. When Lindbergh electrified the world by his flight from New York to Paris, almost every newspaper printed a map showing his course across the Atlantic. Many others must have had my experience of trying to explain to friends why he flew so far north. Why should he have been seen over Ireland when he was going to Paris? My longest argument was with a college professor who did not know what the newspaper account meant when it said that Lindbergh followed a great circle. He was not familiar with the technical term "great circle" and seemed to think that for some temperamental reason Lindbergh chose to fly in a huge circular path. Isn't it just as desirable in our modern social life to know the most important sayings of Euclid as it is to recognize the most common quotations from Shakespeare?

3. *Mensuration formulas and methods.* These follow from the geometric facts and relations. Their necessity and utility need no comment.

4. *Cultivation of space perception,* including the representation of three-dimensional objects by two-dimensional drawings. The natural ability to visualize objects and relations in space varies greatly among individuals. There is no doubt that this natural ability can be increased by cultivation, and that such cultivation has important practical results as well as less tangible effects on general mental development.

**Why Solid Geometry is on the Decline.** With all the cogent reasons that can be advanced for the study of geometry, why is it that the most valuable part of the subject, namely, solid geometry, is on the decline? Why have all colleges that are not technical schools ceased to require it? Why have more and more schools and colleges withdrawn courses in it so that more and more students have no opportunity to study it? With a large number of students solid geometry is unquestionably just naturally unpopular. To an appreciable number of teachers solid geometry is distasteful. To the general public, including some professional mathematicians, the study of solid geometry in preparatory school appears to be futile.

Now I believe this situation exists because of the common meaning attached to solid geometry. It brings up a vision of meticulous proofs of obvious facts, of the devil's coffin and other complicated figures called by more or less meaningless names, of long and involved series of steps to establish a conclusion which often is only vaguely understood. This unfortunate conception is due largely to us, to our understanding of what we should teach. It seems particularly regrettable because the subject can be made extremely interesting. This is not my individual opinion alone, but that of many others who have studied the situation and have asked what is to be done about it. The answer is that we must change the character of the course. We must give up some of the less valuable features and replace them by others.

**Solutions Offered by the College Entrance Examination Board.** Two solutions of the problem have been offered by the College Entrance Examination Board:

1. The requirement labeled Mathematics D. Solid geometry.
2. The requirement labeled Mathematics cd. Plane and solid geometry. Minor requirement.

**Mathematics D.** It is my impression that comparatively few people understand the intent of the Mathematics D requirement, and to make it plain let me explain some of the history of its formulation.

The present requirement was formulated by a commission of eleven members appointed by the College Entrance Examination Board in 1920. Four of these members had served also on the National Committee appointed by the Mathematical Association of America to make recommendations concerning the content of courses in mathematics in the secondary schools. At the first meeting of the Commission a resolution was presented which was the outgrowth of the work of the National Committee. I can not reproduce the wording of the resolution but it was to the following effect:

*a.* That sufficient drill in formal geometric proofs is given in plane geometry and that it can be very largely dispensed with in solid geometry.

*b.* That mensuration be given much more prominence.

*c.* That much more work be done to develop space perception, relations of objects in three dimensions, and the representation of these relations by drawings.

The resolution met with no opposition and received enthusiastic support from some members of the body. It was agreed, however, that such a radical change would have to be made gradually and, as the result of considerable discussion, it was decided to formulate two requirements to be known as Solid Geometry A and Solid Geometry B, the first to be the traditional course and the second a new one defined along the lines suggested in the resolution. The idea was expressed in the meeting that the proposed new requirement would gradually make its way in the schools and ultimately replace the traditional one so that finally the so-called A requirement could be discontinued.

The work of the Commission was apportioned among various subcommittees on algebra, geometry, trigonometry, etc., with a special committee on Solid Geometry B. The work of the general committee on geometry consisted in drawing up a syllabus for each of the requirements in geometry. Its chief contribution was the selection of certain starred propositions chosen for their importance and suitability for examination purposes, with the idea that less effort should be devoted to memorizing the whole list of propositions and more time should be given to originals. The task of the special committee on Solid Geometry B consisted in selecting a small list of propositions and in drawing up a description of the new type of work to be introduced. The result of its attempt appears as the Appendix, pages 35-43, of Document No. 108, which contains the definition of the requirements of the College Entrance Examination Board.

When the different subcommittees reported to the Commission, it appeared that there was less difference between the A and B requirements than had been expected. Both reports recommended essentially the same amount of geometric knowledge, a curtailment of the time spent on book propositions, and an increase in the time spent on originals and applications. The essential difference was that the A requirement included eight starred propositions in Book VI and the B requirement included none. Nothing was more natural than to attempt to combine the two statements into one. The final statement covering this combination is as follows:

> The Board wishes to accord all due latitude in the treatment of the subject of solid geometry. It recognizes the value of the further training in logical demonstration which supplements the study of plane geometry and is given in standard courses at the present time. It recognizes also that the intuitive geometry of the early school course may well be carried further as regards both a firmer grasp on space relations and the visualization of space figures, and the mensuration of surfaces and solids in space.

> The examinations will be constructed with reference to this larger interpretation of the requirement. In the past, the candidate has been expected to answer six questions, and this will be assumed for convenience in defining the nature of the new examinations. These papers will consist of seven questions, of which the candidate will be expected to answer six. Two of these questions will call for demonstrations of propositions from the starred list, but not both of these propositions will be chosen from Book VI. Many teachers have felt that the amount of formal demonstration demanded by this Book has been excessive and has obscured the subject matter. The purpose of the new requirement is to give the teacher a freer hand, enabling him, if he so desires, to teach the facts concerning the relations of lines and planes in space by means of problems and constructions.

**What Can Be Done.** The question now is, What can a teacher do who wishes to break away from the drudgery of forcing unwilling pupils to reproduce book proofs and substitute material which appeals to the imagination, which stimulates the curiosity of the pupil to find out for himself results which are not announced in the statement of the problem, and still prepare the pupils to meet the College Entrance Examination Board requirement? Such a teacher can omit a formal proof of every one of the thirty-six propositions in Book VI. I have done it with college freshmen and believe the result is good. It is merely necessary that the content of the book be understood.

The relations between lines and planes in space can easily be shown by a few simple models constructed extemporaneously from two or three books and pencils. Very few of the facts require proof. If any one is not accepted readily by the whole class, it should have an informal but convincing demonstration.

An explanation of the meaning of the propositions should then be followed by many simple problems making use of them. A single example will illustrate the idea. What is the angle between two diagonals of a cube? Most pupils will answer that immediately. With all the symmetry involved what could the answer be but 90°? And they will be much surprised, when the problem is analyzed and worked out, to find that the answer is 70° 32′. It is an excellent introductory problem both in the steps of the solution and in the unexpected result. The introduction of numerical trigonometry in elementary algebra opens up a great quantity of material in both plane and solid geometry which could not be touched before, and probably very few of us are taking advantage of this liberation from the domination of the 30°-60° right triangles. Further applications and developments are suggested, but by no means exhausted, in the Appendix of the Board's pamphlet. This procedure has the advantage of saving time and sustaining interest. It avoids the danger of disgusting the pupil by proofs of facts that anyone can see and thus killing his interest at the start.

**Mathematics cd.** The second solution of the problem of geometry teaching offered by the College Entrance Examination Board is the requirement known as "Mathematics cd. Plane and Solid Geometry. Minor requirement." The suggestion of such a course was made to the Commission by Professor Dunham Jackson, who came to the first meeting just after the faculty of the University of Minnesota had voted to eliminate solid geometry from the curriculum. He was anxious that something be done to retrieve the loss but felt that we could probably not get more than one year for geometry in our preparatory schools. He suggested a course which should include both plane and solid geometry, but which should require no more time than is usually given to plane geometry. The proposal went through without opposition, because no one would be required to take the course and it seemed quite right to offer the opportunity to anyone who desired it.

**Present Status of the Two Requirements.** The progress of the new geometry under the Mathematics D requirement can not readily be measured, and we can only surmise that little has been accomplished. As to the Mathematics cd requirement we need only turn to the Secretary's reports to assure ourselves that it has not been used. During the past four years the number of books in geometry written by the Board's candidates was 24,432. Of this number seventy-two were in Mathematics cd. After we have been told by the readers that most of the seventy-two gave unmistakable evidence of not knowing enough geometry to pick the right question paper, we are fairly safe in saying that no progress has been made with the minor requirement.

The minor requirement has recently had some able champions in Professor Beatley and Professor Tyler, and a committee is now at work to promote the idea. Until more progress has been made in the development of the course, it is quite natural that the Board should feel that it has done all that it can. A definition of the requirement has been formulated and examinations are set every year. Any candidate who chooses may take the examination. Any teacher or school may prepare for it—and any college may accept it. The colleges have had no opportunity to express their attitude, because the question has not been put to them in the only form which they can answer, namely, by candidates offering the subject. Personally, I have not the slightest doubt that the minor requirement in plane and solid geometry would be accepted everywhere, except perhaps in some technical schools, in place of the plane geometry alone.

**Importance of Experimental Work.** Who, then, has the opportunity and responsibility of changing the situation in geometry? It seems to narrow down to the schools. Any school, or any teacher who is given a free hand in a school, has the opportunity to teach a course in solid geometry which is radically different from the traditional one, a course which can not fail to arouse some curiosity, which appeals to pupils as worth while and to administrators as having practical value. Such a course has been taught successfully, is approved by the Board, and is accepted by every college in the country. We need only textbooks a little better adapted to the purpose, a little bolder in breaking with tradition.

If the longer course proves impossible in some cases, the next best chance is the minor requirement. I feel so sure that there will be no difficulty in having such a course accepted by the colleges that it seems unnecessary to consider this phase as an obstacle. The course, however, will have to be developed. Some enthusiastic believers in the subject, with opportunities to experiment with classes and time to work out details, must show the rest of us how it can be done and prepare the texts with which to do it. There is presented here an opportunity for valuable constructive work, for I can not believe that the subject will continue to be divided into plane geometry and solid geometry. Considering present needs, it seems more advantageous to develop the simpler parts of three-dimensional geometry simultaneously with the corresponding work in two dimensions.

Our young people of to-day have such a variety of interests that it is quite natural for them to become impatient with a subject which appears to have little, if any, connection with ordinary existence. They can not be censured for feeling that every effort must be directed toward something that will count, not necessarily toward greater earning capacity, but at least toward greater mastery of the problems of present-day civilization which they are just beginning to sense. With the growing importance of mathematics to an increasing number of people it seems more than ever unfortunate if there is any step in the mathematical ladder which arrests progress. The traditional course in solid geometry often presents such an obstacle. When a pupil does not study solid geometry, it usually means that there is a break of at least a year before he can go on with further work in mathematics. He not only misses the content of the course and the increased power that can come from it, but he loses much that he has already gained. As a college freshman he finds himself usually at a great disadvantage in competition with others whose mathematical career has suffered no break, and whose mathematical maturity is greater than a difference of one year's work would indicate.

**Present View of the Situation.** Now it is quite possible and highly desirable that we shall sometime have a rearrangement of the mathematics curriculum which will offer to students in the last year of preparatory school a course which will be more attractive than the present ones and which will better prepare them for continuing the study of mathematics in college. In the meantime, while the pioneers are doing their work, the best single step that we can take is to teach solid geometry from the point of view of the B requirement described above. Under the present definitions of requirements in geometry of the College Entrance Examination Board there lies before us the opportunity of doing a considerable service not only for the single subject of geometry but also for mathematics throughout the whole curriculum of the secondary schools and colleges.

# DEMONSTRATIVE GEOMETRY IN THE SEVENTH AND EIGHTH YEARS

By VERA SANFORD
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**Possibilities of an Early Approach to Demonstrative Geometry.** The development of the course of study in mathematics in the junior high school, involving as it does the early introduction of the ideas of algebra and trigonometry, raises the question of the value of an earlier use of demonstrative geometry. One solution urges that a unit of this subject be taught in the ninth year, the work being somewhat less rigorous than the corresponding part of the traditional course. A second solution involves the informal introduction of the subject in the seventh and eighth years. It is the purpose of this chapter to point out, by describing a specific piece of work, how this second suggestion may be put into practice with the brighter students; not, it must be understood, as a unit set off from the pupil's study of intuitive and experimental geometry, but rather as an integral part of his study of geometric relationships. It is, as he may perhaps grow to realize, another method of thinking about the topics with which he deals.

The objection will be raised that an important reason for presenting the unit of demonstrative geometry in the ninth year is its benefit to the student whose school life terminates at this point, and that such students seldom rank in the upper quartile of their group. This point must be granted; yet experience shows that these students sometimes find great satisfaction in the study of elementary algebra. It is not improbable that something of the same sort may follow from the study of elementary demonstrative geometry even at an earlier point.

In the case of students who meet the subject later, whether in the ninth year or in the tenth or eleventh, their acquaintance with the concepts of geometric reasoning may be quite as valuable as their preliminary knowledge of geometric forms.

**View Based on Experience.** Although the considerations that follow are based on my experiences over a period of nine years in the Lincoln School of Teachers College, they should be considered as mere suggestions of the possibilities of this work rather than as a report of the results of scientific experimentation. During the early part of this time the necessity of some such action was becoming more and more clear to me through my work in the senior high school field where the problem was the building of a suitable course on the basis of the junior high school work. During the latter part of this period, I had an opportunity to see what changes might be effected in the junior high school course to provide for better preparation for the work of the later grades.

**How the Question Arose.** My first interest in this problem was influenced by conditions which I think were not unique. I found that classes trained in junior high school mathematics tended to be restive when they were asked to prove theorems that in an earlier grade they had assumed to be true or that they had established by experiment. Even postulating a liberal number of propositions would not solve the difficulty, for their work had involved congruency, similarity, area formulas, and even the Pythagorean theorem. The essential differences between experimental and demonstrative geometry were emphasized but again and again the students would refer to their former work; for example, in proving that only one perpendicular could be dropped from a point to a line by reference to the formula for the sum of the angles of a triangle—a theorem that had not then been proved. As authority, they cited the page on which the statement appeared in their junior high school text. They had a useful vocabulary of geometric terms, but apparently their study of geometry that was intuitive had inhibited their acceptance of geometry that was demonstrative. A possible remedy was to reduce the amount of intuitive geometry or to change its subject matter, perhaps emphasizing symmetry of different types and paying little attention to congruence and similarity. Yet these very topics were fundamental in parts of the junior high school course that were themselves of intrinsic usefulness.

**Experiment versus Reasoning.** Another way of meeting the problem was to attack it at its source. and to add to seventh and eighth year geometry the consideration of the difference between an opinion based on intuition and one based on reasoning. At this point, I was greatly assisted by a chance circumstance that not only provided a natural opening for the problem with the seventh grade, but suggested a situation that might be duplicated with other classes at the same point in other years. A social studies discussion of the way in which men arrive at a statement of a scientific law was interrupted by the close of the period, and the class came in a body to the mathematics room. Mindful of the close interplay between departments which is characteristic of the school, the social studies teacher suggested that the pupils see if they could not find immediate assistance in mathematics. Fortunately the day's work was well adapted to this problem; so, building on the considerations of the previous hour, we pretended complete ignorance of our previous work in the study of the sum of the angles of a triangle, and started afresh.

**First Step.** The first step was to state the problem: *to construct a triangle whose angles are of given size.* This was clearly a simple matter compared to primitive man's experiments with various remedies for disease or an inventor's groping toward scientific discovery. When we listed the values of the angles which were shown by experiment to permit the construction of a triangle and those that did not, we were clearly in the trial-and-error stage which had been discussed in the social studies class. The study of our successes and failures led to the conclusion that the work was not necessarily possible when just any angles were given, for the drawing of the first two inexorably determined the size of the third which might or might not be the same as the size of the third given angle. Our conclusion was that some connection between the size of the third angle and that of the other two was essential to the drawing of the triangle.

At this point we might well have traced a comparison between the work at hand and the methods of the medicine man and his herbs or the astrologer who based his conclusions on observations of the planets.

Scrutiny of our data showed that although there was wide variation in the size of the angles in the different triangles, the sum of the angles in any one case was in the neighborhood of 180°. Our rule had now reached the stage where it read, "The construction seems to be possible if the sum of the given angles is 180°."

**Second Step.** We then passed to the second stage to see whether this rule of thumb would work with cases chosen at random. The class argued that a single case in which the rule did not yield a satisfactory result would be sufficient to nullify it, and it clearly would be impossible to draw *all* possible cases. It was here, then, that we undertook the time-honored tests of cutting out a triangle and piecing the corners together. We turned a ruler through each angle of the triangle in turn and were convinced that it had made half of a complete turn (rotation). Finally, we accepted the conclusion that the sum of the angles of a triangle drawn in a plane is 180°. We had recapitulated the early steps in the discovery of a scientific law: first, trial and error leading to the formulation of a rule of thumb; and second, the tests of this rule. The final step of reasoning from previously accepted hypotheses to laws was as yet untouched.

**Third Step.** The next day, however, a member of the class asked if the sum of the angles of a four-sided figure was also a definite amount. Robert, brilliant but hasty, said, "Yes, 360°, for a rectangle is a four-sided figure and it has four right angles."

"But," objected Edson, "not all four-sided figures are rectangles. Suppose it were a parallelogram?"

"That's all right," said Robert, "what you squeeze out at one corner comes in at the other."

Edson's reply indicated lack of conviction in these blanket statements, and Margaret interpolated the opinion that some four-sided figures were, as she put it, "neither rectangles nor parallelograms but cockeyed." With that she drew an irregular quadrilateral on the board. Cecilia (I. Q. 106) then said, "That's all right. I'll show you (drawing a diagonal). The sum of the angles here is 180° and here it is 180° and together the sum for the whole thing is 360°."

The class agreed that Cecilia's scheme was neither intuition as was Robert's, nor was it based on measurement. They decided to call it a "proof by reasoning." From that time on, we definitely set ourselves the job of proving our guesses in this manner whenever possible.

In the course of this work, besides getting an inkling of the meaning of demonstration, the class discovered for itself Robert's use of an unproved converse, and criticized him for his tacit assumption that since all rectangles are quadrilaterals, all quadrilaterals are also rectangles. Examples of converse statements were proposed and discussed, and these ranged from very simple ones to the table talk at the Mad Hatter's party with the "I say what I mean is the same as I mean what I say."

**Value of an Early Approach to Geometry.** This was the writer's first introduction to the informal use of the basic idea of a proof by reasoning in the seventh grade, but it suggested a line of thought to which other experiences were naturally allied. Why should not these students early become accustomed to reasoning from previously accepted theorems or from postulates? Not, of course, with the technical vocabulary of "theorems" and "postulates," but with the concepts informally expressed? Why should they not learn the pitfalls of the glib quoting of a converse to a true statement? Why should they not learn how to phrase the opposite of a given statement? The work to which they come in science and the social studies, for example, endeavors to stimulate independent thought. It offers many opportunities for reasoning from hypotheses but it also offers many occasions in which the pupil is likely to fall into habits of incorrect reasoning in the matters of converses and opposites mentioned above. Is the seventh or the eighth grade too soon to begin to build an appreciation of a proof and its implications?

**Repetition of This Piece of Work.** It not infrequently happens that the opportunities that come by chance in one class can be induced by suggestion in another. This has been the case in the instance cited here, and in other classes it has been a simple matter to progress from the proof of a theorem by measurement to the proof of a corollary of this theorem by reasoning. Other features have entered naturally—the arbitrary nature of a definition, the idea that a postulate is accepted without proof, and the contrast between a direct proof and a proof by exclusion.

It has been my experience that these considerations make for a greater unity in the geometry of the junior high school, that they provide opportunity for individual work of high quality, and that they bridge the gap between two opposing types of geometry—informal and demonstrative.

# A UNIT OF DEMONSTRATIVE GEOMETRY FOR THE NINTH YEAR

By JOSEPH B. ORLEANS
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**Purpose of the Unit.** A unit of demonstrative geometry, to cover in time the equivalent of six to eight weeks, can best fit into the work of the ninth year if algebra is begun in the eighth year and if a foundation of intuitive geometry is laid in the seventh year. Since many of the pupils in the algebra classes continue with plane geometry the tenth year, nothing must be presented in this unit which will interfere with the work of that year. The purpose of this course is to give the pupils a notion of what a logical proof is. It is not necessary, therefore, to emphasize the translating of word problems into geometric language. That may well be left for the next grade. The work need concern merely the relationship between the facts that are given, the conclusion to be reached, and the steps one must follow to reach the conclusion. Since the unit is to be part of the work of a year which is devoted also to algebra and to the solution of the right triangle, and is to be spread over at least the 9B grade, it would be well to correlate the geometry, as far as possible, with the algebra and the trigonometry.

**Axioms and Postulates.** The axioms and the necessary postulates are not listed in the outline which follows because they will appear in the work one at a time in various connections and the individual teacher must emphasize and teach them as they occur.

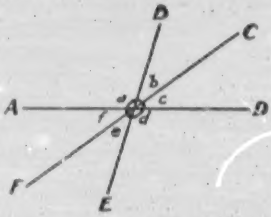
**Nature of the Unit.** This plan takes the form of a series of exercises which are based upon certain postulated geometric facts and which lend themselves to logical demonstration, special emphasis being laid upon certain important problems.

## Outline of the Unit

I. **PRELIMINARY DEFINITIONS.** Line, point, angle (straight, right, obtuse, acute), vertical angles. In connection with these the class should review the fundamental constructions which lead to the postulating of the congruence theorems.

II. Vertical Angles Are Equal. An informal proof. Pupils always wonder why they must prove that vertical angles are equal when they can readily see that they are equal. An informal conversation about a pair of vertical angles and their common supplement will help to establish the fact without dependence merely upon appearances and will also give the pupils the beginning of the relationship between statement and reason leading to a conclusion.

(a) *Numerical exercises.*



(1) $\angle b = 20°$, $\angle c = 60°$, $\angle AOE = ?$
(2) $\angle BOF = 130°$, $\angle c = 40°$, $\angle a = ?$
(3) $\angle FOD = 140°$, $\angle b = 60°$, $\angle a = ?$
(4) $\angle f = 60°$, $\angle b = 25°$, $\angle d = ?$
(5) $\angle f = b°$, $\angle d = 100°$, $\angle b = ?$
(6) $\angle a = 2\angle c$, $\angle e = 60°$, $\angle c = ?$
(7) $\angle COA = 140°$, $\angle EOC = 120°$, $\angle DOB = ?$

(b) *Proofs.* (Use the above diagram.)

(1) $\angle b + \angle c = \angle AOE$.
(2) $\angle BOF - \angle c = \angle a$.
(3) $\angle BOF - \angle f = \angle d$.
(4) $\angle f + \angle b + \angle d = \angle 180°$.
(5) $\angle COA + \angle DOB + \angle EOC = 360°$.
(6) $\angle COA + \angle EOC - \angle AOE = 2\angle a$.
(7) If $\angle f = \angle e$, then $\angle b = \angle c$.

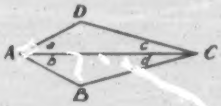
(c) *More difficult exercises.* (Use the above diagram.)

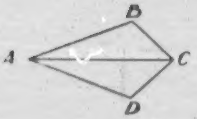
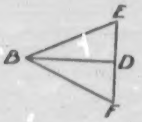
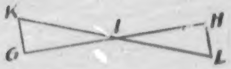
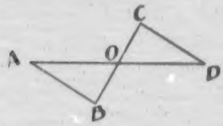
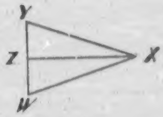
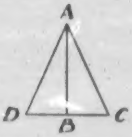
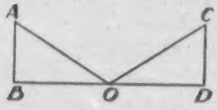
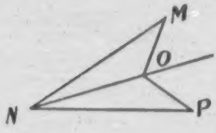
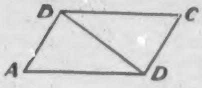
(1) $\angle COA = 150°$, $\angle EOC = 130°$, $\angle a = ?$
(2) $\angle BOF = 140°$, $\angle COA = 125°$, $\angle d = ?$
(3) $\angle AOE + \angle COB = 140°$, $\angle c = 40°$, $\angle e = ?$
(4) If $\angle BOF = \angle FOD$, prove $\angle f = \angle e$.

(d) *Geometric application.* If a straight line bisects one of two vertical angles, it bisects the other one also.

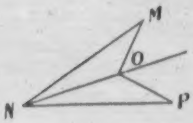
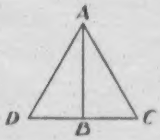
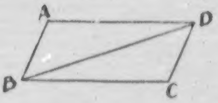
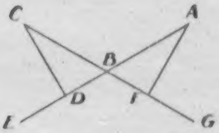
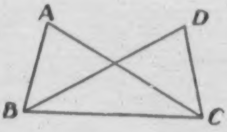
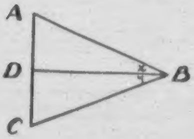
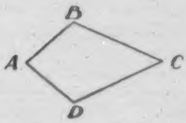
III. The Congruence Theorems. The congruence of pairs of triangles based upon *s. a. s.* = *s. a. s.*, *a. s. a.* = *a. s. a.*, and *s. s. s.* = *s. s. s.* by means of triangles constructed with given parts and cut out of paper or cardboard.

***Exercises***

| | Given | Prove |
|---|---|---|
| (1)  | $\angle a = 30°$,<br>$\angle b = 30°$,<br>$\angle c = 20°$,<br>$\angle d = 20°$. | $\triangle ABC \cong \triangle ADC$. |

| | | Given | Prove |
|---|---|---|---|
| (2) |  | $AC$ bisects $\angle A$.<br>$AC$ bisects $\angle C$. | $\triangle ABC \cong \triangle ADC$. |
| (3) |  | $BD$ bisects $\angle B$.<br>$EF$ is $\perp$ to $BD$. | $\triangle BED \cong \triangle BDF$. |
| (4) |  | $I$ is the midpoint of $GH$.<br>$KG$ is $\perp$ to $GH$ and $HL$ is $\perp$ to $GH$. | $\triangle KGI \cong \triangle HIL$. |
| (5) |  | $AD$ and $BC$ are straight lines.<br>$AO = OD$ and $BO = OC$. | $\triangle AOB \cong \triangle COD$. |
| (6) |  | $XY = XW$.<br>$XZ$ bisects $\angle X$. | $\triangle XZY \cong \triangle XZW$. |
| (7) |  | $DBC$ is a straight line.<br>$AB$ is $\perp$ to $DC$ and $DB = BC$. | $\triangle ABD \cong \triangle ABC$. |
| (8) |  | $AB$ is $\perp$ to $BD$ and $CD$ is $\perp$ to $BD$.<br>$O$ bisects $BD$.<br>$AB = CD$. | $\triangle ABO \cong \triangle OCD$. |
| (9) |  | $MN = NP$,<br>$MO = OP$. | $\triangle MNO \cong \triangle NPO$. |
| (10) |  | $BC = AD$,<br>$AB = CD$. | $\triangle ABD \cong \triangle BCD$. |

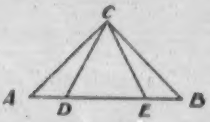
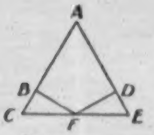
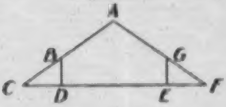
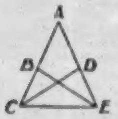
Teach the fact of the equality of the corresponding sides and of the corresponding angles of congruent triangles. Use exercises 1 to 10 for drill in proving corresponding parts equal.

| | | *Given* | *Prove* |
|---|---|---|---|
| (11) |  | $NO$ bisects $\angle PNM$.<br>$MN = NP$. | $\angle M = \angle P$. |
| (12) |  | $DBC$ is a straight line.<br>$AD = AC$,<br>$\angle DAB = \angle BAC$. | $\angle D = \angle C$. |
| (13) |  | $AB = CD$,<br>$\angle DBA = \angle BDC$. | $AD = BC$.<br>$\angle BAD = \angle BCD$.<br>$\angle CBD = \angle ADB$ |
| (14) |  | $CD$ is $\perp$ to $AE$,<br>$AF$ is $\perp$ to $CG$,<br>$BD = BF$. | $\angle C = \angle A$. |
| (15) | Same diagram as Ex. 14. | $CB = BA$,<br>$DB = BF$,<br>$AE$ and $CG$ are straight lines. | $CD = AF$. |
| (16) |  | $AB = DC$,<br>$AC = BD$. | $\angle A = \angle D$. |
| (17) |  | $AB = BC$,<br>$\angle x = \angle y$. | $\angle A = \angle C$. |
| (18) |  | $AB = AD$,<br>$BC = CD$. | $\angle B = \angle D$. |

(19) If two sides of a triangle are equal, the angles opposite those sides are equal.

(Introduce the need of a construction line to help one to reach the conclusion. Emphasize the importance of drawing the construction line with a purpose in mind.)

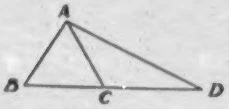
*Exercises Based on the Isosceles Triangle Exercise*

| | | *Given* | *Prove* |
|---|---|---|---|
| (20) |  | $AC = BC$,<br>$AD = EB$. | $CD = CE$. |
| (21) | Same diagram as Ex. 20. | $AC = BC$,<br>$AE = DB$. | $CD = CE$. |
| (22) |  | $F$ is the midpoint of $CE$.<br>$AC = AE, BC = DE$. | $BF = DF$. |
| (23) |  | $AC = AF$.<br>$BD$ is $\perp$ to $CF$.<br>$GE$ is $\perp$ to $CF$.<br>$CD = EF$. | $BD = GE$. |
| (24) |  | $AC = AE$.<br>$B$ is the midpoint of $AC$.<br>$D$ is the midpoint of $AE$. | $CD = BE$. |

(25) If the three sides of a triangle are equal, the three angles are equal.

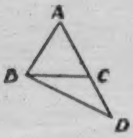
Postulate the fact that the exterior angle of a triangle is greater than either opposite interior angle. The proof of this theorem does not fit into a series of exercises like those in this outline. It is impossible to expect the pupils to suggest the necessary construction lines. They must get them either from the teacher or from a textbook. There is, therefore, no loss in postulating this theorem.

(26)



*Given:* $AB = AC$.
*Prove:* $\angle B$ greater than $\angle D$.

(27)

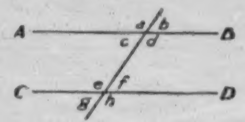


*Given:* $AD$ greater than $AB$.
$AC = AB$.
*Prove:* $\angle DBA$ greater than $\angle ADB$.

IV. **Parallel Lines.** *Two lines are parallel if a transversal makes a pair of alternate interior angles equal.* (This theorem introduces the pupils to a simple form of the indirect proof.)

*Exercises*

(28) Tell why lines $AB$ and $CD$ are parallel if



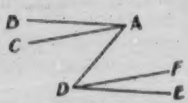
(a) $\angle c = 70^\circ$ and $\angle f = 70^\circ$.
(b) $\angle c = 60^\circ$ and $\angle e = 120^\circ$.
(c) $\angle a = 110^\circ$ and $\angle f = 70^\circ$.
(d) $\angle b = 60^\circ$ and $\angle f = 60^\circ$.
(e) $\angle a = 120^\circ$ and $\angle g = 60^\circ$.

(29) Prove that $AB$ is ∥ to $CD$ if $\angle b = \angle f$. (Use diagram in Ex. 28.)

(30) Prove that $AB$ is ∥ to $CD$ if $\angle a = \angle h$. (Use diagram in Ex. 28.)

(31) Prove that $AB$ is ∥ to $CD$ if $\angle b = \angle g$. (Use diagram in Ex. 28.)

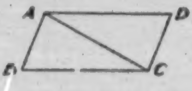
(32)



*Given:* $\angle CAD = \angle FDA$ and
$\angle BAC = \angle EDF$.
*Prove:* $AB$ ∥ to $DE$.

(33) Use diagram of Ex. 32. *Given:* $\angle BAD = \angle ADE$ and
$\angle BAC = \angle EDF$.
*Prove:* $AC$ ∥ to $DF$.

(34)



*Given:* $AB = DC$ and
$\angle CAB = \angle DCA$.
*Prove:* $AD$ ∥ to $BC$.

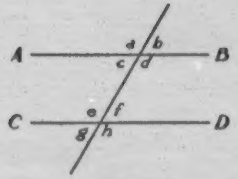
(35) Draw two lines $AB$ and $CD$ bisecting each other at $E$. Prove $AC$ ∥ to $DB$.

(36) Two lines perpendicular to the same line are parallel.

Postulate the fact that *if two lines are parallel, the alternate interior angles are equal.*

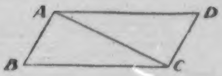
(37) *If two lines are parallel, the corresponding angles are equal.*

(38)



*Given:* $AB \parallel$ to $CD$.
(a) If $\angle c = 70°$, $\angle f = ?$
(b) If $\angle c = 80°$, $\angle e = ?$
(c) If $\angle c = 75°$, $\angle g = ?$
(d) If $\angle a = 110°$, $\angle f = ?$
(e) If $\angle b = 80°$, $\angle g = ?$
(f) If $\angle h = 100°$, $\angle b = ?$

(39)

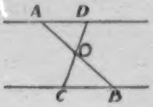


*Given:* $AD = BC$ and $AD \parallel$ to $BC$.
*Prove:* $\angle DCA = \angle BAC$.

(40) Same diagram as Ex. 39.

*Given:* $AB \parallel$ to $CD$ and $AD \parallel$ to $BC$.
*Prove:* $\angle B = \angle D$.

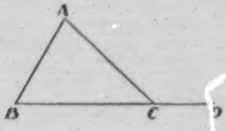
(41)



*Given:* $AD \parallel$ to $CB$ and $AO = OB$.
*Prove:* $DO = OC$.

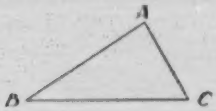
(42) If a line is perpendicular to one of two parallel lines, it is perpendicular to the other one also.

(43)



In the accompanying diagram, prove that $\angle DCA = \angle A + \angle B$. (Since we are comparing the exterior angle with two other angles, divide the exterior angle into two parts in the most convenient way. Let the class discuss the various possibilities and suggest the construction line.)

(44) Prove that $\angle A + \angle B + \angle C = 180°$.



(The pupils have learned this fact in their intuitive geometry. Since we are comparing the three angles with a straight angle, what is the best way to introduce a straight angle into the diagram?)

*Exercises Based on Ex. 44*

(45) If $\angle A = 70°$ and $\angle B = 50°$, $\angle C = ?$

(46) How large is each angle of an equiangular triangle?

(47) The vertex angle of an isosceles triangle is 30°. How large is the base angle?

(48) The base angle of an isosceles triangle is 35°. How large is the vertex angle?

(49) What is the number of degrees in each angle of an isosceles right triangle?

(50) Two angles of a triangle contain respectively 60° and 67°. What is the size of the third angle?

(51) Two angles of a triangle contain respectively $x^\circ$ and $y^\circ$. What is the size of the third angle?

(52) Two angles of a triangle are equal and the third angle contains $2k$ degrees. How many degrees are there in each of the equal angles?

(53) If the three expressions $2x$, $3x - 2$, and $x - 8$ represent the number of degrees in the three angles of a triangle, find the number of degrees in each angle.

(54) Two angles of a triangle are in the ratio of 9 to 7 and the third equals the difference between the other two. Find each angle.

(55) Find the number of degrees in the angles of a triangle, if the first is 12° more than the second and the third is double the second.

(56) Find the number of degrees in the angles of a triangle, if one angle is twice the second and three times the third.

(57) Prove that the sum of the angles of a quadrilateral is 360°.

(58) Explain why a triangle can have no more than one right angle.

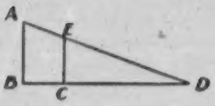
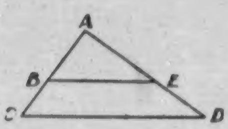
(59) Explain why a triangle can have no more than one obtuse angle.

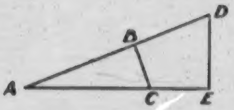
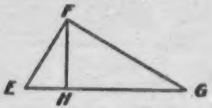
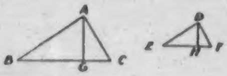
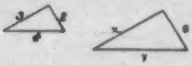
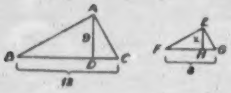
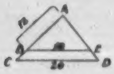
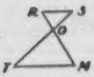
(60) If two angles of a triangle are equal respectively to two angles of another triangle, the third angle of the first is equal to the third angle of the second.

V. Similar Triangles. Definition. Illustrate by means of parallel lines. Contrast similar and congruent triangles. Contrast equal corresponding sides of congruent triangles and proportional corresponding sides of similar triangles. Relate this work to the solution of the right triangle in the trigonometry.

Postulate the fact that *two triangles are similar if at least two angles of one are equal to two angles of the other.*

*Exercises Based on Similar Triangles*

| | | *Given* | *Prove* |
|---|---|---|---|
| (61) |  | $EC$ is $\perp$ to $BD$ and $AB$ is $\perp$ to $BD$. | $\triangle ECD \sim \triangle ABD$ and write a proportion. |
| (62) |  | $BE$ is $\parallel$ to $CD$. | $\triangle ABE \sim \triangle ACD$ and write a proportion. |

| | | *Given* | *Prove* |
|---|---|---|---|
| (63) |  | *DE* is ⊥ to *AE* and *CB* is ⊥ to *AD*. | $\triangle ABC \sim \triangle ADE$ and write a proportion. |
| (64) |  | *EFG* is a right angle.<br>*FH* is ⊥ to *EG*. | $\triangle EFH \sim \triangle EFG$ and write a proportion. |
| (65) | Use the diagram of Ex. 64. | *EFG* is a right angle.<br>*FH* is ⊥ to *EG*. | $\triangle FGH \sim \triangle EFG$ and write a proportion. |
| (66) |  | $\triangle ABC \sim \triangle DEF$.<br>*AG* is ⊥ to *BC* and *DH* is ⊥ to *EF*. | $AG:DH = AB:DE$. |
| (67) |  | The two triangles in the diagram are similar. | Find the values of $x$ and $y$. |
| (68) |  | The two triangles in the diagram are similar.<br>*AD* is ⊥ to *BC* and *EH* is ⊥ to *FG*. | Find the value of $x$. |
| (69) |  | *BE* is ∥ to *CD*. | Find the value of *AC*. |
| (70) |  | *RS* is ∥ to *TM*.<br>*RM* is a straight line.<br>*TS* is a straight line. | $\triangle ROS \sim \triangle TOM$.<br>Write a proportion based on this fact. |

(71) If a pole 120 ft. high casts a shadow of 40 ft., how long is the shadow of a pole 30 ft. long?

**Summary.** In the above outline the pupil becomes acquainted with the meaning of a geometric demonstration through the notions of congruence (in connection with which the teacher may introduce symmetry), parallelism, and similarity. These are the three main threads that run throughout the plane geometry of the tenth year. With some classes some teachers may be able to go a bit further and develop the Pythagorean theorem, which will follow directly from the exercises dealing with similar triangles. This can very well be correlated with the solution of the incomplete quadratic equation in algebra. No attempt has been made to suggest the methods that the teacher is to use in introducing the work or in developing the various parts. The outline merely suggests what might be the content of a unit of demonstrative geometry in the ninth year.

# A UNIT OF DEMONSTRATIVE GEOMETRY FOR THE NINTH YEAR

By JOSEPH SEIDLIN

*Alfred University, Alfred, N. Y.*

**Nature of the Unit.** The unit of demonstrative geometry here suggested is based in principle upon the recommendation of the National Committee on Mathematical Requirements*: *"E. Demonstrative Geometry.* The demonstration of a limited number of propositions, . . . the principal purpose being to show to the pupil what 'demonstration' means."

That a pupil may *see* a demonstration he must understand every step in it and he must be satisfied with the genuineness of the structure as a whole. But he need not necessarily be able to reproduce the "proof" synthetically or "get at it" analytically.

Other things being equal, the "demonstration" in this unit is no less rigorous than that in present-day tenth year geometry. Though assumptions are made rather freely, the extreme of assuming "everything and anything" is avoided. The writer feels that too many axioms and postulates would complicate the unit and destroy, to a proportionate extent, the validity of demonstration.

Practical applications, basal theorems, the development of spatial imagination are unquestionably desirable objectives in the teaching of tenth year geometry. But they have exercised very little influence in the choice of the content of this unit. It is distinctly not a mere preparatory course to further work in geometry. On the other hand, this unit is so constructed that it will neither interfere with nor inhibit the more formal geometry of the senior high school.

**Axioms, Postulates, and Definitions.** At the very outset the pupil must learn to distinguish between assumptions in demonstrative geometry and inferred facts in intuitive or experimental geometry. For instance, in informal geometry we assume the equality of vertical angles because the angles "look equal" and as a result of a simple experiment we assume that the sum of the interior angles of a triangle is 180°; in demonstrative geometry we may assume the equality of vertical angles irrespective of "looks" or "experiment." In other words, the pupil must be given to understand that the axioms and postulates, the definitions, and the "given" part of the theorem are indisputable authorities in demonstrative geometry.

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* *The Reorganization of Mathematics in Secondary Schools*, edited by John W. Young. Published by Houghton Mifflin Company.

In the plan of this unit the assumptions and definitions immediately precede their actual use in proofs. Varying pupil conditions may require a richer list of exercises and, likely, other assumptions and definitions. In no sense is this unit to be regarded as a complete or "closed" whole.

### Introductory Group

**Assumptions.** The following assumptions will be necessary:

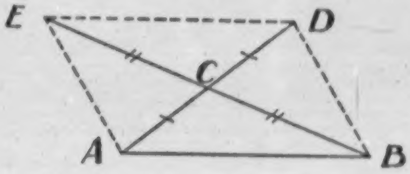
1. *When two straight lines intersect, the vertical angles are equal.*
2. *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the triangles are congruent.*

**Definitions.** We shall also need the following definitions:

1. The equal parts of congruent triangles are called their *corresponding* parts.
2. *Distance* between two points is measured along the straight line joining the two points.

**Exercises.** The following exercises may then be given:

1. If in $\triangle ABC$, $AC$ is prolonged to $D$ so as to make $CD = AC$, and $BC$ is prolonged to $E$ so as to make $CE = BC$,



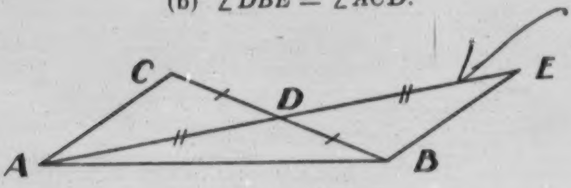
we can then prove that

(a) $\triangle CDE \cong \triangle ABC$.
(b) $E$ is as far from $D$ as $A$ is from $B$.
(c) $\triangle ACE \cong \triangle BCD$.
(d) $E$ is as far from $A$ as $D$ is from $B$.

These four exercises may be proved informally either in the order given or in the order a, c, b, d.

2. If in $\triangle ABC$, $D$ is the midpoint of $BC$ and through $D$ we draw $AE$ so as to make $DE = AD$, and if we then draw $BE$, we can prove that

(a) $\triangle DBE \cong \triangle ADC$.
(b) $\angle DBE = \angle ACD$.



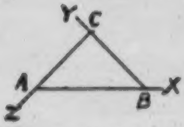
### First Group

**Assumptions.** The following assumptions will now be necessary:

1.1* *The whole is greater than any of its parts.*
2.1 *In any function any quantity may be substituted for its equal.*

**Definitions.** We shall also need the following definition:

1.1 In the triangle $ABC$,
angles $ABC$, $BCA$, and $CAB$ are its *interior* angles;
angles $CBX$, $ACY$, $BAZ$ are called *exterior* angles.



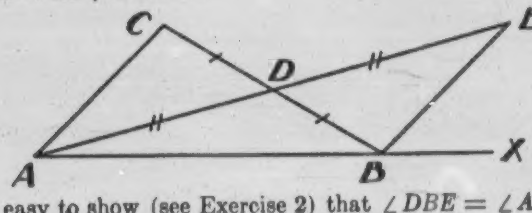
**Theorem I.** The following theorem may then be proved:

*An exterior angle of a triangle is greater than either of the opposite interior angles.*

*Given.* In $\triangle ABC$, with $AB$ produced to $X$, $CBX$ is an exterior angle and $\angle ACB$ is one of the opposite interior angles.

*Prove 1.* $\angle CBX > \angle ACB$.

*Proof.* Through the midpoint of $BC$, say $D$, draw $AE$ so as to make $DE = AD$; draw $BE$.



It is easy to show (see Exercise 2) that $\angle DBE = \angle ACB$. But we know that $\angle CBX > \angle DBE$ (see Assumption 1.1). And, therefore, $\angle CBX > \angle ACB$ (see Assumption 2.1).†

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* Read "First assumption in the first group" and similarly for the other notation like this.

† The reasons should be stated in words, of course.

*Prove 2.* $\angle CBX > \angle CAB$.

The proof may be developed completely and independently of (1), or it might suffice merely to indicate the two new elements.

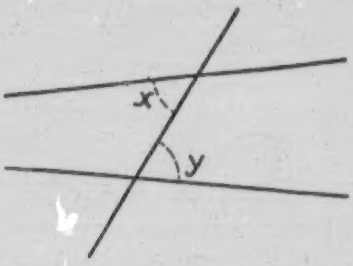
## Second Group

**Assumptions.** Another assumption will now be necessary.

1.2 *Three mutually intersecting lines, not passing through the same point, form a triangle.*

**Definitions.** This definition is needed:

1.2 When two lines are cut by a third (called a transversal) the angles formed on the alternate sides of the transversal and "inside" the two lines are called *alternate-interior* **angles.**

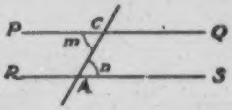


Angles $x$ and $y$ are alternate-interior angles.

**Theorem II.** The following theorem may now be proved.

*Two lines cut by a third will not meet, however far produced, if the alternate-interior angles are equal.*

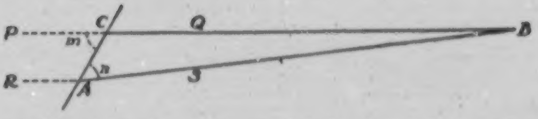
*Given.* Lines $PQ$ and $RS$ are cut by the transversal $AC$ so that $\angle m = \angle n$.



*Prove.* $PQ$ cannot meet $RS$.

*Proof.* At first it may seem (to the pupil) that it is impossible to prove this theorem since, after all, it is not practicable to keep on producing these lines forever. Here; again, we have the opportunity to glorify the power of logical demonstration.

We are going to prove that *because* $\angle m = \angle n$ it is impossible for $PQ$ to meet $RS$ anywhere: What sort of figure would be formed by $AC$ and $PQ$ and $RS$ if the latter two met?



Obviously a triangle *ABC* would be formed (see Assumption 1.2). What is the relation of $\angle m$ to $\angle n$? (see Definition 1.1). It would therefore follow that $\angle m > \angle n$ (see Theorem I). But *we know* that $\angle m = \angle n$ (the only fact "given"). This rules out the possibility of the "exterior-opposite-interior" angle relationship between angles *m* and *n*. In other words, *AC*, *PQ*, and *RS* cannot possibly form a triangle. And, therefore, *PQ* cannot meet *RS* (see Assumption 1.2).

**Definition.** Another definition will now be necessary:

2.2 *Two lines in the same plane that do not meet, however far produced, are called parallel lines.*

**Exercises.** We may now prove these three exercises:

1.2 Restate Theorem II, using Definition 2.2.
2.2 Any number of well-known corollaries and exercises may follow Theorem II. (The choice is left to the instructor.)
3.2 Two sides of a triangle are prolonged their own lengths through the vertex of the triangle. Prove that the line joining their ends is parallel to the base.

## Third Group

**Definitions.** The following definitions will now be needed:

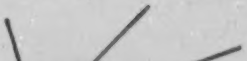
1.3 Straight angle.
2.3 Sum of angles about a point.
3.3 Exterior angles of any polygon.
4.3 *n*-sided polygon.
5.3 Exterior-interior, or corresponding, angles.

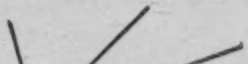
**Assumptions.** Two more assumptions must now be made:

1.3 The more common axioms.
2.3 If two parallel lines are cut by a transversal the corresponding angles are equal.

**Exercises.** The following exercises can now be proved:

1.3 *If two angles have their sides parallel, left to left and right to right, the angles are equal.*
2.3 The extension of Exercise 1.3 to more than two angles:

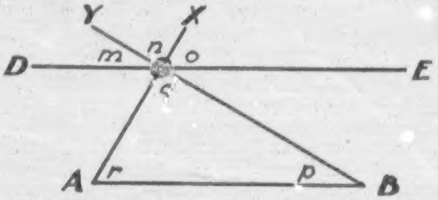




3.3 By means of the accompanying diagram show that the sum of the interior angles of a triangle is equal to a straight angle.

We have $AC$ produced to $X$.
$BC$ produced to $Y$.
$DE \parallel$ to $AB$.

Show that angles $m, n, o$ are equal, respectively, to angles $p, q, r$.

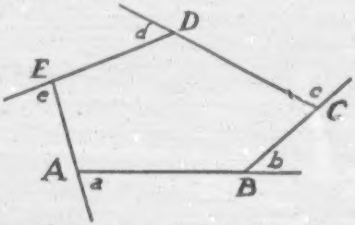


**Theorem III.** The following theorem should then be given:

*The sum of the exterior angles of any polygon is equal to two straight angles.*

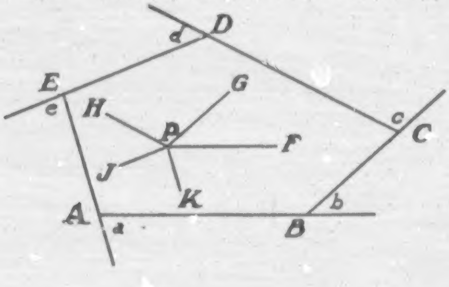
Since "an $n$-sided polygon" cannot be drawn (to the complete satisfaction of the pupil), we will employ a five-sided polygon and indicate toward the end of the proof that this in no way impairs the generality of the theorem.

*Given.* Polygon $ABCDE$ with its sides produced so as to form the exterior angles $a, b, c, d, e$.



*Prove.* $\angle a + \angle b + \angle c + \angle d + \angle e = 2$ straight angles.

*Proof.* Through some point, such as $P$, within polygon $ABCDE$, draw $PF \parallel$ to $AB$, $PG \parallel$ to $BC$, $PH \parallel$ to $CD$, $PJ \parallel$ to $DE$, $PK \parallel$ to $EA$.



$$\left.\begin{aligned}\angle a &= \angle KPF\\ \angle b &= \angle FPG\\ \angle c &= \angle GPH\\ \angle d &= \angle HPJ\\ \angle e &= \angle JPK\end{aligned}\right\}\text{See Exercise 2.3.}$$

$$\angle a + \angle b + \angle c + \angle d + \angle e = \angle KPF + \angle FPG + \angle GPH + \angle HPJ + \angle JPK \text{ (Sums of equals are equal).}$$

But $\angle KPF + \angle FPG + \angle GPH + \angle HPJ + \angle JPK = 2$ straight angles (Definition 2.3).

$\therefore \angle a + \angle b + \angle c + \angle d + \angle e = 2$ straight angles (Assumption 2.1).

In what way would the proof be altered by the choice of a polygon of six sides, or eight sides, or ten sides, or, conceivably, any number of sides?

**Exercises.** The following exercises can now be proved:

4.3 Simple numerical exercises involving exterior angles of equiangular polygons.

5.3 Show that in any polygon $A = 180° - a$ (where $A$ is any interior angle and $a$ is its adjacent exterior angle).

6.3 The sum of the interior angles of a polygon of five sides equals three straight angles.

7.3 The sum of the interior angles of a polygon of six sides equals four straight angles.

8.3 The sum of the interior angles of a triangle is a straight angle. (Note that the proof here is independent of that of Exercise 3.3.)

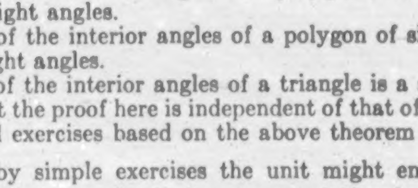
9.3 Numerical exercises based on the above theorem and exercises.

Enriched by simple exercises the unit might end here. Certainly not much more should be attempted if only three or four weeks are allotted to this work. Under a more favorable time apportionment, say six weeks, it would be desirable to enlarge the unit by including the following propositions on areas. In the latter case it would be necessary to have enlarged the scope of exercises on parallelograms. In particular, we need to have considered:

(1) *Opposite sides of a parallelogram are equal.*

(2) *A diagonal divides a parallelogram into two equal triangles.*

**Theorem IV.** We may now introduce the following theorem: *If a parallelogram and a rectangle have the same base and altitude, their areas are equal.*



*Given.* $ABCD$, a parallelogram with base $AB$ and altitude $BE$.
$ABEF$, a rectangle with base $AB$ and altitude $BE$.

*Prove.* Area of $ABCD$ = Area of $ABEF$.

*Proof.* Area of $ABCD$ = Area of $ABED$ + Area of $\triangle BCE$
(The whole is equal to the sum of its parts.)
Area of $ABEF$ = Area of $ABED$ + Area of $\triangle ADF$
(The whole is equal to the sum of its parts.)
(i.e., the parallelogram and the rectangle have $ABDE$ in common; all we need to show, therefore, is that $\triangle BCE = \triangle ADF$.)

In $\triangle$ $BCE$ and $ADF$
(1) $BC = AD$ (Opposite sides of a parallelogram are equal.)
(2) $BE = AF$ (Opposite sides of a parallelogram are equal.)
(3) $\angle EBC = \angle FAD$ (Exercise 1.3)
(4) $\therefore \triangle BCE = \triangle ADF$ (Assumption 2)

Consequently, Area of $ABED$ + Area of $\triangle BCE$ = Area of $ABED$ + Area of $\triangle ADF$ (Sums of equals are equal.) Area of $ABCD$ = Area of $ABEF$. (Quantities equal to equal quantities are equal.)

**Exercises.** These exercises can now be given:

1.4 *If two parallelograms have the same base and altitude, their areas are equal.*

2.4 *If two triangles have the same base and altitude, their areas are equal.*

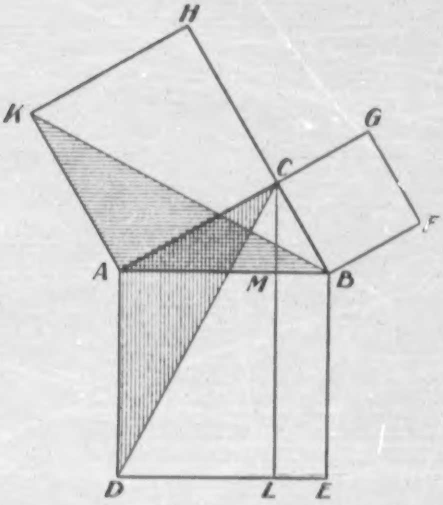
3.4 *If a triangle and a rectangle have the same base and altitude, the area of the triangle is half the area of the rectangle.*

**Theorem V.** The Pythagorean theorem can now be presented:

*The square on the hypotenuse of a right triangle is equal to the sum of the squares on its sides.*

*Given.* Right $\triangle ABC$ with $BC \perp$ to $AC$
Square $ADEB$ on $AB$ (hypotenuse)
Square $BFGC$ on $BC$
Square $ACHK$ on $AC$

*Prove.* Area of square $ADEB$ = Area of square $ACHK$ + Area of square $BFGC$.



*Proof.* The added lines in the above diagram are $CL$ drawn perpendicular to $DE$; $CD$ and $BK$.

$CL$ divides the square $ADEB$ into two rectangles—$ADLM$ and $LEBM$. We will develop the proof by showing that

(1) Area of $ADLM =$ Area of square $ACHK$
(2) Area of $LEBM =$ Area of square $BFGC$

To prove (1) we show that

(3) $\triangle ABK = \triangle ADC$
(4) Area of $\triangle ABK = \frac{1}{2}$ Area of square $ACHK$
(5) Area of $\triangle ADC = \frac{1}{2}$ Area of rectangle $ADLM$

To prove (3) we show that

(6) $AB = AD$ (sides of square $ADEB$)
(7) $AK = AC$ (sides of square $ACHK$)
(8) $\angle KAB = \angle CAD$ (each $= 90° + \angle CAB$)

To prove (4) and (5) we use Exercise 3.4, since $\angle BCH = 90° + 90° = 180°$, and, therefore, $BCH$ is a straight line parallel to $AK$.

To prove (2) we draw the additional lines $AF$ and $CE$ and then follow the procedure of the proof of (1).

The above proof is partial, sketchy, and very informal. Nevertheless it is conclusive and satisfying. For the brightest and most interested pupils the complete formal proof may be given. The duller pupils may be spared even the informal proof. To the bright pupil the Pythagorean theorem geometrically proved may seem a fitting "grand finale."

**Exercises.**

1.5 Any desirable number of exercises involving the Pythagorean theorem.

**Conclusion.** The purpose of this unit, it will be recalled, is "to show to the pupil what 'demonstration' means." The propositions were expressly so chosen as to make the introduction to logical proof most real, most palatable, and least painful to the average pupil. The unit of demonstrative geometry will serve its purpose best if it leaves with the pupil a pleasant and lasting impression that there is a subtle, mentally-satisfying quality in "proof by reasoning" not found either in "proof by experiment" or in "intuition."

# TEACHING PLANE AND SOLID GEOMETRY SIMULTANEOUSLY

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**Introduction.** Under the pressure of change in secondary mathematics, plane and solid geometry have been compressed into a one-year course. Such a combination entails two marked changes: first, the elimination of about one-third of the content, and second, the simultaneous introduction of the more fundamental parts of plane and solid geometry.

**Reduction of Content.** Teachers of secondary mathematics have realized for some time that they have been attempting to teach material which rightly belongs only in advanced courses especially planned for teachers of mathematics. Consider the following: the proofs of the congruent triangle theorems, incommensurable cases, the geometric proof of the Pythagorean theorem, and much of the material on limits. Postulating the three congruence theorems on triangles and solving the Pythagorean theorem algebraically strengthens rather than weakens the pupil's power. Certainly it helps to eliminate difficulty.

**Simultaneous Introduction of Plane and Solid Geometry.** How can plane and solid geometry be taught simultaneously? The sequence of Euclid must be largely followed, but we should not forget that the geometry of Euclid was constituted and arranged for mature minds—not for children of the age now found studying geometry in secondary schools. Probably the first step in this new presentation is the enlargement of definitions.

**Enlargement of Definitions.** Take, for instance, the concept "angle." Why limit the child to plane angles? Let him see dihedral and polyhedral angles at the same time. He is daily faced by the dihedral angle when his book is opened, when a piece of paper is folded, and when he sees the intersection of any two walls in his home, in his schoolroom, or elsewhere. Moreover, he sees trihedral angles in the corners of boxes, the corners of rooms; and he discovers polyhedral angles in roofs of houses, in church steeples, and in the crystals of various minerals. Does isolation of special angles contribute more to their meaning than consideration of them simultaneously? Probably there has been a tendency toward too much separation in the past.

Think of the possibilities in the following: perpendiculars to lines and to planes, plane and spherical triangles, bisectors, complementary and supplementary angles, circles and spheres, and parallel lines and parallel planes. Spatial concepts and plane concepts contribute much to each other, but since we live in a three-dimensional world the spatial concepts should be easier of comprehension. In our zeal to follow the past, however, we have turned psychological order around.

**Congruency of Triangles.** Now let us examine some of the ways in which solid geometry may be introduced with plane geometry. Take the three fundamental congruence theorems on triangles. Is there any real reason why the triangles involved may not be revolved from one plane to another or why we may not draw oblique lines from any point in a perpendicular to a plane? Indeed, there seems to be no real danger in letting the pupil know that there are other types of triangles—spherical triangles, for example. We try to make our pupils "plane minded" instead of "space minded"—thus we begin to isolate and disconnect the various phases of mathematics. The marvel is that pupils have ever attained a high degree of understanding and power in this subject.

**Parallel Lines.** The theorems in plane geometry based on parallel lines and parallel lines cut by transversals are not especially difficult for pupils to understand, but parallel planes and parallel planes cut by transversal planes seem to annoy and disturb them. I see no reason for separating the treatment of these theorems. If two parallel lines cut by a third line are pulled through space in a direction perpendicular to them, the student can visualize the straight lines, tracing out two parallel planes, cut by a third plane. Moreover, the plane angles formed by the original lines trace out dihedral angles. Since plane angles measure dihedral angles the story is told convincingly.

Just as the ideas connected with a parallelogram grow out of the theorems concerned with pairs of intersecting parallel lines, so also do those connected with parallelopipeds grow out of the situation where pairs of parallel planes intersect. Similarly, polygons and prisms are related. Pull a polygon through space and the prism can be visualized.

**Volumes and Areas.** Areas of plane figures and areas and volumes of three-dimensional figures have so long been separated that the ordinary student finds it difficult to understand much about them.

**Circles.** One of the most interesting opportunities for teaching plane and solid geometry simultaneously is found in associating the circle and the sphere. Revolve a circle about one of its diameters as an axis and observe the path traced by an arc, a chord, a radius, and a central angle. The association clarifies each.

**Loci.** Probably no topic offers richer material for bringing together the content of plane and solid geometry than the discussion of loci. The locus of a point equidistant from two parallel lines and from two parallel planes, the locus of a point equidistant from the sides of a plane angle and also from a dihedral angle, and the corresponding two- and three-dimensional associates based on the locus of a point equidistant from the extremities of a given line, together with countless other cases, indicate the marvelous opportunities available.

Possibly a small unit of solid geometry will remain at the end of the course—a unit which seems to be more effective when taught alone. However, with experience, more and more of solid geometry will become absorbed in plane geometry. As a result, the teaching of geometry will become more dynamic and the understanding of it less difficult for most pupils.

# AN EXPERIMENT IN REDISTRIBUTION OF MATERIAL FOR HIGH SCHOOL GEOMETRY

By GERTRUDE E. ALLEN
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**Introduction.** A distinguished Scotchman, visiting professor at the University of California, while addressing a Phi Beta Kappa gathering recently referred to the hordes that throng our campus as a "university proletariat innocent of the vaguest suggestion of culture or scholarship—yet withal, friendly, capable, nice young things." To initiate a European guest more completely in the profession of public education in a democracy, we should introduce him to the multitudes which overflow our high schools, the source of a selected group which forms the so-called "university proletariat."

The management of these "friendly, capable, nice young things" is a very insistent and vital problem to the adults concerned. It means nothing in their young lives for their elders to stand aloof and condemn them in casual review. Ralph Waldo Emerson once said, "The secret of education lies in respecting the child." As a high school teacher, I champion the cause of high school children. Our greatest service to them is to preserve and foster their breezy freshness, originality, independence, and enthusiasm; to direct their *purposeful activity* in order that it may be efficient and significant in the world in which they live. In the attempt to do this the natural questions which the thoughtful teacher is always asking himself are—Why? What? When? How? The specific question opened for discussion here is—In what way are the growth and development of senior high school children best promoted and nurtured through the study of elementary geometry? The purpose of this chapter is to describe in some detail the modified program of tenth grade geometry at the University High School of Oakland, California—a program which has been in the process of being evolved by experimentation over a period of about ten years in an attempt to redistribute the material and to adapt the methods of presentation better to the needs of the children entrusted to our care. The statements preceding the outline are introduced by way of explanation and defense.

You have already a visitor's impression of our lively foreground. The background in California high schools requires perhaps a little explanation. First, public schools are of far greater significance and importance than private schools and California spends money lavishly on the equipment and maintenance of the public high school. Superior private schools are comparatively few in number and have in many cases largely a non-resident clientele. The State Board of Education defines requirements for graduation and it must approve public high school courses of study. The Board does not prescribe uniform textbooks nor set state examinations. No mathematics is required for high school graduation. Graduates of accredited high schools may be admitted to the University of California without examination—subject to approval of the admissions committee. A new ruling, effective in 1931, transfers certain responsibilities from the high school principal to this committee. The university committee requires that a candidate for matriculation offer elementary algebra and plane geometry. Further than that, technical colleges require four years of mathematics as a prerequisite for university courses. There is always the alternative of passing the College Entrance Board Examinations; the point of immediate interest is, however, that elementary algebra and plane geometry are required as preparation for college and that the university is the goal of a large majority of our young people. There is a reasonable degree of freedom for every teacher of geometry to exercise his initiative and judgment; in the last analysis, however, his students must make good when they continue their work and he must justify his goals and standards to his principal and superintendent so that the percentage of failures is within the range of normal expectancy. The teachers set or select their own examinations, and these examinations are only one of the factors determining promotion; only in exceptional cases do our students take the College Entrance Board Examinations. The high schools and the individual teachers are rated by the university on the basis of the success with which their graduates carry on at the university. Such a plan seems eminently just—it offers the greatest possible stimulus to teachers to work with the welfare of the children as the ideal always uppermost in their minds.

**Criticism of the Teaching of Geometry.** As a matter of background, I think it is worth while also to take stock of some of the significant things that educational leaders have said about geometry and about the way it is taught. A few criticisms and suggestions are given in the following quotations:

> "Schemes of geometrical education . . . are lacking in foundation, method and extent. Euclid's Scheme—itself utterly unsuitable as an introduction to the subject, has been so far tampered with that hardly any scheme remains. So long as no attempt is made to devise a connected development based on the many intuitions which are common to all civilized beings before they reach maturity—so long will the subject realise a painfully small proportion of its potential value."—G. St.L. Carson, *Essays on Mathematical Education.*
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> "As a school subject, geometry is capable of improvement in spirit and in content. In most schools there has been a good deal of memorizing of demonstrations—original exercises played a negligible part without purpose or pleasure."—David Eugene Smith, *First Yearbook,* National Council of Teachers of Mathematics.
>
> "The chances for development of mental power outside of geometry are much greater today than they were before the dawn of the present age of science. . . . We have many good geometricians who seem to possess small logical sense in other affairs. . . . logical power is a growth and usually a slow growth—an attribute of maturity and not of youthfulness. . . . An average pupil can memorize and reproduce anything that is printed in the geometries. We deceive ourselves into believing that the pupils were really comprehending the thing which they seemed to be doing. . . . Humanize the teaching of high school geometry. . . . Admit that geometry should be taught for the benefit of the students, and that this benefit consists quite as much in the inward development of power to use the subject in its manifold applications as in the outward insistence upon its theoretical aspects for the purpose of mental discipline."—H. E. Slaught, *Humanizing of High School Mathematics.*
>
> "Euclid's brilliant success in organizing into a formally deductive system the geometric treasures of his times has caused the reign of science in the modern sense to be so long deferred. . . . The learner is led blind-folded. . . . Method of investigation is concealed. . . . It is essential that the boy be familiar by way of experiment, illustration, measurement, and by every possible means, with the ideas to which he applies his logic and, moreover, that he should be interested in the subject."—Perry.
>
> "It is to the interest of the geometrician himself that he be persuaded to look about for ways of cultivating space imagery which are more efficient than those afforded in the logical courses now offered in demonstrative geometry. If demonstrative geometry is to be made a training in reasoning which may be used in other fields, there must be radical changes in the methods of teaching it."—C. H. Judd, *Psychology of High School Subjects.*

"Our geometric concepts have been reached for the most part by purposeful experience. . . . Geometry has sprung from interest centering in spatial relations of physical bodies. . . . It is wrong in elementary geometry to cultivate predominately the logical side of the subject and neglect to throw open to young students the wells of knowledge contained in experience."—Mach, *Space and Geometry.*

"Mathematics generally, and particularly geometry, owes its existence to the need which was felt of learning something about the relations of real things to one another. . . . According to axiomatics, the logical-formal alone forms the subject matter of mathematics. . . . It is not surprising that different persons must arrive at the same logical conclusions when they have already agreed upon the fundamental laws as well as the methods by which these laws are deduced therefrom. . . . It is clear that the system of concepts of axiomatic geometry alone cannot make any assertions as to the relations of real objects. . . . To be able to make such assertions, geometry must be stripped of its merely logical-formal character by coördination of real objects of experience with the empty framework of axiomatic geometry. We need also to add the relation of solid bodies of three dimensions. . . . Geometry thus completed is evidently a natural science. . . . We may in fact regard it as the most ancient branch of physics. . . . I attach special importance to the view of geometry which I have just set forth because, without it, I should have been unable to formulate the theory of relativity."—Albert Einstein, *Sidelights on Relativity.*

"The reasoning we have to depend on in the daily conduct of life is almost all probable and not demonstrative."—C. W. Eliot.

"Very few of the activities of the mind are thinking—but no school work is worthy of a place in a school program that does not require some thinking. . . . Thinking is the activity of a person dominated by a particular purpose. Essentials of thinking—something to think about, a motive for thinking, a method of thinking, inherent capacity to think at all. . . . Reasoning begins not with premises but with difficulties. . . . Difficulty, perplexity, demand for the solution of a problem . . . steadying and guiding factor in the entire process of reflection. . . . What is important is that the mind should be sensitive to problems and skilled in methods of attack and solution."—John Dewey, *How We Think.*

## Course of Study in Elementary Geometry: *

**A. One-Year Course.** Tenth school year, or later in special cases by advice of school counselor.

Prerequisite—Elementary Algebra.
Required for admission to the University of California.

**B. Aims.** The following aims are to be achieved.

1. The Development of Space Intuition by:
   a. *"Laying a foundation of experience upon which to build."*
      (1) Experiment and measurement.

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* Reprinted from Oakland Course of Study, Bulletin 77.

    - (2) Constructions.
        - (*a*) Geometric drawing.
        - (*b*) Making models and crude instruments.
    - (3) Observation of geometric forms in nature, architecture, and decorative design.
    - (4) Exercise of spatial imagination.
- *b*. "*Organizing* a *body* of *knowledge out* of this *experience*." The definite goal is:
    - (1) To gain an accurate knowledge of the significant propositions of geometry.
    - (2) To develop and learn for practical use the essential formulas of mensuration.
    - (3) To reveal possibilities in further exploration and to find incentives to carry on.
- *c*. "*Applying the resulting knowledge* to *practical* use in the concrete world."

2. To Furnish Favorable Material for Exercise in the Process of Logical Thinking.
    - *a*. To develop an understanding and appreciation of the method of deductive reasoning in the field of geometry.
    - *b*. To form habits of exact, truthful statement, and of logical organization of ideas in this field.
    - *c*. To establish and exercise a conscious technique of thinking—using as a basis Dewey's analysis of a complete act of thought:
        - (1) A felt difficulty.
        - (2) Its location and definition.
        - (3) Suggestions of possible solution.
        - (4) Development of reasoning of the bearings of the suggestions.
        - (5) Further observation and experiment leading to its acceptance or rejection.
    - *d*. To foster all possible transfer of ability to the solution of new problems, both mathematical and non-mathematical.

C. Outline of Course.

1. Congruence and Equality.
    - *a*. Congruent triangles—their significance in the study of trigonometry and in mechanical constructions.
    - *b*. Sum of angles of triangle and polygon.
        - (1) Importance in theory of trigonometry.
        - (2) Isosceles and equilateral triangles, regular polygons.
        - (3) Applications in design for surface covering, such as quilt patterns, tiling, carpet, and oilcloth patterns.
    - *c*. Right triangle and theorem of Pythagoras.
        - (1) Deriving theorems for length of side opposite an obtuse and an acute angle in a triangle.

(2) Different interesting proofs and various important applications of the Pythagorean theorem.
(3) Ratio of side to diagonal of a square; of altitude to base in an equilateral triangle.

2. Similarity and Symmetry.
   a. Similar triangles and similar polygons; the right triangle; sine, cosine, and tangent functions of an angle; problems in heights and distances; ratio of lines in right triangle formed by dropping a perpendicular from the vertex of the right angle to the hypotenuse and the constructions depending upon the equality of these ratios.
   b. Ratio of corresponding lines, corresponding areas, and the volumes of similar solids.
   c. Symmetry—central, axial, planar. Observations of symmetry in nature, architecture, stage settings, art design, room furnishings.
3. Form and Position.
   a. Rectilinear figures and solids bounded by planes.
      (1) Intersecting lines and intersecting planes; pencil of planes; concurrent lines in a triangle; the centers of a triangle and the related circles.
      (2) Perpendicular lines and perpendicular planes;—rectangles and rectangular solids.
      (3) Parallel lines and parallel planes with their related angles; parallelograms and parallelopipeds.
   b. Circles.
      (1) Subtended arcs, angles, and chords.
      (2) Secants and tangents to a circle with related angles, relation of segments of intersecting lines which meet in a circle.
      (3) Regular polygons inscribed and circumscribed; limiting value for ratio of perimeter to diameter; computing $\pi$ (the ratio of the circumference to the diameter); computing area of regular polygons and circles; informal treatment of theory of limits.
   c. Locus of a point which moves about in space, but satisfies fixed conditions; locus of the moving point when positions are restricted to the plane; informal treatment of variables and constants.
4. Geometric Construction.
   Constructions of more or less complexity to stimulate thought in applying previously acquired knowledge in new situations.
5. Mensuration.
   a. Development of standard formulas for areas and volumes.
   b. Numerical computations involving these formulas applied in miscellaneous problems of practical value.

## MAKING SOME SOLID GEOMETRY AN INTEGRAL PART OF TENTH YEAR WORK IN ELEMENTARY GEOMETRY

**First Method.** We find that for children the most natural and interesting introduction to the abstract phase of geometry is through their concrete experiences in their real world of three dimensions; and we also find that conditions most favorable to transfer of training obtain in the application and operation of general principles to problems in this same world of three dimensions. Certain fundamental concepts are more effectively developed and investigated by generalized treatment than by a restricted treatment which limits the excursions of thought to flatland—such concepts, for example, as congruence, symmetry, similarity, locus, geometric forms, angles, parallelism, perpendicularity, motion by rotation and translation, position referred to $X$–$Y$–$Z$ axes, size, and measurement of length, area, volume, and angles.

The fundamental principles of congruence are more significant if they are explained in connection with solids as well as plane figures, reference being made to the importance of exact congruence in the economics of industry. Children contribute a remarkable variety of material for illustrations—radio parts, Ford parts, telephones, quart milk bottles, standard screws, electric base plugs, lead pencils, Mason fruit jars, linoleum patterns, tablet armchairs, and the like. Similarity may be illustrated by an object and a miniature model of it, preferably two objects which can be measured and weighed to compare corresponding lengths, areas, and weights. A satisfactory treatment of similar figures should bring to the student the conviction that when two similar solids are compared, two corresponding lines have the same ratio as any other two corresponding lines, corresponding areas have the same ratio as the squares of any two corresponding lines; and *also* that the *volumes* have the same ratio as the *cubes* of two corresponding lines. Symmetry may be illustrated by the human body—for example, the two hands—or it may be illustrated by a pair of shoes, a vase, a shapely tree, a well designed building, a balanced stage setting—as well as an isosceles triangle, a kite, or a fleur-de-lis pattern. Parallelism is better understood by observing parallel walls in the corridor, the floor and ceiling of the room, a picture molding and the ceiling, parallel rows of trees in an orchard, parallel columns in a beautiful building, than by parallel lines drawn on a sheet of paper. The nature of a locus is made clearer by numerous examples of restricted motion in space and the corresponding result when the locus is limited to a plane. The tridimensional locus problems are entirely intuitive but children never question this procedure; they probably will challenge the arbitrary requirement of a logical proof for any theorems on loci. The type of generalized treatment indicated in the preceding illustrations serves to engage the attention and interest of the beginner, enriches the field of his geometric experience, and motivates rather than distracts his efforts in logical demonstration.

**Second Method.** A second means of unifying plane and solid geometry is the extension of certain facts and theorems of the first to include analogous facts and theorems of the second. For example, from whatever aspect one chooses to discuss definitions for point, line, surface, and solid, magnitudes of one, two, three dimensions are involved, and some child with a fertile imagination, active curiosity, or good memory for hearsays will wish to pursue the discussion into the realm of the fourth dimension. Postulating the statement that a straight line is the shortest distance between two points—how can we account for Lindbergh's route from New York to Paris? What is the shortest air line route from New York to San Francisco, from San Francisco to Honolulu? Extend the concept of an angle between two lines to include the definition of an angle formed by a line and a plane and an angle formed by two planes. Illustrate the carpenter's test for making an upright perpendicular to the plane of the floor. Note the one-to-one correspondence of parallel lines cut by a transversal line and parallel planes cut by a transversal plane; parallelogram and parallelepiped; regular polygons and regular convex polyhedra; area of a rectangle and volume of a rectangular parallelepiped; area of a parallelogram and volume of an oblique parallelepiped; area of a triangle and volume of a triangular pyramid; area of a circle and volume of a sphere; perigon and steregon; radian and steradian; central angle in a circle and intercepted arc and polyhedral angle at center of a sphere and intercepted spherical polygon; area of a trapezoid and volume of the frustum of a pyramid. Rotation about an axis is a useful device in a student's thinking: what solid is generated by rotating a rectangle about its base? by rotating a right triangle about either arm of the right angle, about the hypotenuse? a circle about its diameter? an ellipse about its major axis, about its minor axis? Other analogies will suggest themselves and may be introduced at the discretion of the teacher when by doing so interest in the work is increased or the mathematical outlook widened.

**Third Method.** A third means of lifting demonstrative geometry out of flatland is to select varied positions for the planes in which our figures are drawn—not limiting this to the blackboard or the writing pad. Consider the angle of elevation of the top of the flagpole, the diagonal of the classroom, the total surface of the teacher's desk, the lateral surface of a pyramid, the total surface of the Platonic bodies, the similar polygons formed by parallel transverse sections of a pyramid, and the like. Apply the theorem —"The sum of the squares on the four sides of a parallelogram equals the sum of the squares on the two diagonals"—to different planes in a parallelepiped, and arrive at the conclusion that the sum of the squares on the twelve edges equals the sum of the squares on the four diagonals. There was a good exercise of this description in a recent College Entrance Board Examination: "$V$-$ABCD$ is any pyramid with a rectangular base $ABCD$. Prove that the lateral edges are connected by the relation

$$(VA)^2 + (VC)^2 = (VB)^2 + (VD)^2."$$

**Fourth Method.** The fourth suggestion is to devote the last month of the year's work to mensuration. A teacher can make judicious selection, substitution, and elimination of theorems so that a tenth year class can work very satisfactorily through a course in demonstrative geometry in nine months, including a review and one of the standardized achievement tests. According to our records, eighty per cent of these students take no more work in mathematics. In our judgment, they will get the greatest educational value by rounding out their work with significant applications and practice in accurate and efficient computation. The twenty per cent who continue need such exercise and training even more; furthermore, they will have an opportunity to supplement the tenth year geometry course with a second course in the senior year. We emphasize in mensuration the development of the standard formulas for areas and volumes of solids, the use of these formulas in *thinking through* a problem and setting up the complete plan of solution in equation form, and correct, efficient methods of computation. A good collection of sensible, significant problems is made and there is a preliminary requirement of indicating solutions in equation form, followed later with estimating, computation, and check. Attention is given incidentally to the fundamental principles of perspective drawing. I believe that any open-minded teacher who tries this plan once will never be satisfied unless he continues to follow it.

## MAKING SOME ELEMENTARY WORK IN TRIGONOMETRY AN INTEGRAL PART OF THE GEOMETRY COURSE

**Congruence.** In the treatment of congruent triangles, we may arouse curiosity and stimulate a forward-looking interest if we explain that it is the primary business of trigonometry or triangle measurement to "solve triangles," that is, to compute the unknown elements in a triangle when a sufficient number of elements are known to determine the triangle. Hence the importance of knowing all the different possibilities—or understanding that the different cases may be expressed in the one general statement—a triangle is determined if any three independent elements are given (except in the ambiguous case).

**Similarity.** Particular emphasis may be given to the theorem: "Two right triangles are similar if an acute angle of one equals an acute angle of the other." This fact forms the basis for a large group of important ratios fundamental to the solution of triangles. We form the triangle of reference for a given angle by dropping a perpendicular from the terminal arm of the angle upon the initial arm. Hence the ratio of any two of the lines in a right triangle so formed remains constant for a given value of the angle and changes in value with a change in the size of the angle. These six ratios are called the trigonometric functions of the angle in question. Problems in the solution of right triangles, some practice in the use of the tables of natural functions contained in the geometry textbook, and specializing on sine, cosine, and tangent can be given.

**Projection.** The definition of the projection of a line segment on a coplanar line is $p = l \cos A$. This definition may be used in a very neat proof of the Pythagorean theorem given as supplementary or alternative work; thus:

Given the right triangle $ABC$, right angled at $C$, $a'$ the projection of $a$ on $c$, and $b'$ the projection of $b$ on $c$.

Then

$$c = a' + b'$$

$$c = a \cos B + b \cos A = a \cdot \frac{a}{c} + b \cdot \frac{b}{c}.$$

Hence

$$c^2 = a^2 + b^2.$$

**Law of Cosines.** By using the previous definition of projection the generalized case of the Pythagorean theorem may be extended to read $c^2 = a^2 + b^2 - 2ab \cos C$. It may be applied in solving problems, particularly with angles of 30°, 45°, 60°, 120°, 135°.

**Law of Sines.** The interesting relation between the sides of a triangle and the angles opposite them may be developed through the definition of the sine function. This is significant; indeed, we are challenged to explain it, as children almost invariably jump to the erroneous conclusion that the ratio of two sides of a triangle is the same as the ratio of the two angles opposite. This may be an inference from the case of the isosceles triangle, but it is a very common error and should be corrected. Particular attention should be paid to the ratio of the sides in the two draughtsman's triangles.

**Mensuration of the Circle.** In this unit the pupils usually make "pi-books," containing history and selected problems, and develop the numerical value of $\pi$ by means of the sine and tangent ratios for the inscribed and circumscribed regular polygons, respectively. We supply each with a mimeographed sheet containing eight-place tables for the angles concerned. This is a substitute for the textbook computation and we find that the pupils take great pride in their own theses.

## Making Some Algebra and Arithmetic an Integral Part of the Work in Geometry

**Unification.** Whenever it is practical to do so, we make algebra and arithmetic an integral part of the work in geometry. Aside from the sense of continuity that the student feels, there is an opportunity to exercise fundamental skills and there is a specific advantage in facilitating work in geometry. A free use of symbolic notation facilitates geometric analysis and the expression of it; the use of the formula and equation is essential to the application of theorems proved, or to the process of making geometric knowledge function effectively in action, whichever way one may please to state it. To illustrate: In the triangle *ABC*, designate angles by capital letters and sides opposite by corresponding small letters. Make a practice of indicating line segments by small letters, angles by capitals or Greek letters, using a subscript if several angles have a common vertex. Use the conventional abbreviations in writing out authorities. A few little devices of this sort help the student to keep his "eye on the ball." The equation and formula are fundamental in the work of mensuration. We insist on thinking through the problem and indicating in equation form the final result before any computations are made. As far as numerical computations are concerned, I think we have a right to insist that economical and correct work is an essential part of geometry. It is disconcerting to a teacher, when a class as a group has developed very beautifully the problem of the Golden Section, to discover, by means of the questions asked by those who do not understand, that their difficulties arise from adding fractions by adding both numerators and denominators, indiscriminate canceling, or inability to do the simplest factoring.

**Algebraic Symbolism.** The following are a few specific cases where algebraic symbolism is used:

1. To find the sum of the exterior angles of a polygon of $n$ sides made by producing each of its sides in succession. Let $e_1$ and $i_1$ represent one exterior angle and the adjacent interior angle; let $E$ and $I$ represent the corresponding sums.

   Take the consecutive vertices, $e_1 + i_1 = 1 \text{ st } \angle$

   $e_2 + i_2 = 1 \text{ st } \angle$

   Since this happens $n$ times, $E + I = n \text{ st } \triangle$

   But $I = n \text{ st } \triangle - 2 \text{ st } \triangle$

   Therefore, $E = 2 \text{ st } \triangle.$

2. In the proof for dividing a line segment $l$ in Golden Section internally, $x$ being the larger part, we have from the construction

   $$\frac{l+x}{l} = \frac{l}{x}.$$

   The goal is to prove

   $$\frac{x}{l} = \frac{l-x}{x}.$$

   How can you get in one step what you want from what you have?

3. Somebody brought in this clipping from the Oakland *Tribune*— Is it true? "Material in the great Pyramid would build a wall 2 ft. wide and 4 ft. high from New York to San Francisco."

4. From another College Entrance Board Examination: Is there enough material in 1 cu. ft. of lead to make 200,000 bullets, ¼ in. in diameter?

5. Get a galvanized bucket from the janitor's closet. Make the necessary linear measurements and compute the number of gallons it will hold. Check by liquid measure.

$$d_1 = 9 \text{ in.};\quad d_2 = 11 \text{ in.};\quad h = 12 \text{ in.}$$

$$\text{No. of gal.} = \frac{1}{3}\pi h\,(r_1^2 + r_2^2 + r_1 r_2) \div 231$$

$$V = \frac{1}{3} \times \frac{22}{7} \times \frac{12}{231} \times \frac{301}{4} = \frac{43}{42} \text{ of 4 gal.}$$

6. Imagine the Great Wall of China torn down and built into a rectangular wall which would encircle the earth at the equator (25,000 mi.) and 3 ft. wide. How high would it be? The Chinese wall is 1,500 mi. long, 15 ft. wide at top, 25 ft. wide at bottom, and 20 ft. high.

$$\text{Prism}_1 = \text{Prism}_2$$

$$3x \times 25000 \times 5280 = 1500 \times 5280 \times \frac{15 + 25}{2} \times 20$$

$$x = 8$$

7. More or less algebraic work is necessary in the simpler problems of proportion and also in the problem of the Golden Section of a line.

### How the Combined Course is Worked Out

**Choice of Material.** We have attempted to arrange the work in logically self-consistent units, choosing the material from what seems to be significant and understandable to the fifteen-year-old child and ignoring the artificial man-made barrier between plane and solid geometry. This plan has been justified in our judgment both in the interest of the large majority who take only two years of high school mathematics and in the continuous growth of those who study advanced geometry with introductory work in the calculus. The dictum of certain very wise teachers is amply vindicated over and over again in our practical experience—it is vital to the learning process that a child be taught something which he wishes to learn, that he understand what he is doing, that he be interested in what he is doing, that this interest be sustained through satisfaction in accomplishment which appeals to him as worth while, and that he have opportunity to use the knowledge he has gained in some purposeful activity.

The subject matter content is characterized by an increased number of postulates, a decreased number of theorems to be proved, a close correlation between plane and solid geometry, the incorporation of some algebra, trigonometry, and arithmetic, and the application of general principles in the solution of practical problems.

It appears permissible and advisable in the teaching of beginners to accept as postulates the statement of all possible intuitions. The conventional assumptions would include: (1) the equality of vertically opposite angles; (2) properties of figures which are evident from symmetry; (3) congruence of figures which can be determined by superposition (discarding superposition as a so-called method of proof); (4) congruence of plane figures which possess line symmetry; (5) the angle properties of parallel lines, namely: If two straight lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel; and the converse, if two parallel lines are cut by a transversal, the corresponding angles are equal; (5) the area of a rectangle is equal to the product of its base by its altitude; (6) the central angle in a circle has the same measure in degrees as its intercepted arc; (7) equal central angles of a circle have equal arcs, and conversely; (8) if while approaching their respective limits two variables are always equal, then their limits are equal.

**Omissions and Substitutions.** Omissions and substitutions must be determined by the textbook one uses and also, of course, by the ability of the class. The text used in Oakland is Smith's *Essentials of Plane and Solid Geometry*. Consistent with an agreement previously made, we omit as theorems in this text those specific statements which we develop intuitively and accept as postulates. Furthermore, we omit the proofs of incommensurable cases given in the supplement, pages 425-428; the algebraic manipulations of a proportion, pages 157-161 (making necessary readjustments in the method of the geometric proofs); the two problems which involve dividing a line segment externally, page 183 and the proof as given on page 240; and pages 192, 196, 197, 228, 229; possibly the constructions on pages 223 and 224. We substitute for the theorem on page 209: $A = \frac{1}{2}ab \sin C$; for proof on page 210 a simpler proof based on the ratio of $\frac{1}{2}bh$ to $\frac{1}{2}b'h'$; and at that point extend the comparison to the ratio of the volumes of two similar solids by finding the ratio of $k(lwh)$ to $k(l'w'h')$. For problems on pages 248 and 254 we substitute the trigonometric ratios suggested in the section of this article on trigonometry. Hero's formula, page 194, is developed with the class, and then memorized and used—pupils are not required to reproduce the proof. In the tenth year course in geometry no attempt is made to give formal proofs of the solid geometry theorems. We aim to develop an understanding of certain fundamental principles, a conviction that these principles are indeed true facts. We aim to develop when needed, as clearly and directly as possible, the fundamental formulas listed on page 498 and provide opportunity for the knowledge of these formulas to function in the analysis and solution of worth-while problems of application.

**Formal Proofs in Solid Geometry.** We reserve formal proofs in solid geometry for the second course in geometry offered in the senior year of high school. Select groups of mature students are able to do more rigorous advanced work with profit. They can prove supplementary theorems of plane geometry and the essential theorems of solid geometry, and yet have ample time in their semester's work for introductory work in the calculus or for elementary analytic geometry. One advantage in the redistribution of subject matter here indicated is that time is provided for extension of electives for the superior student. The development and entire treatment is kept thoroughly concrete. It can be managed in a way to furnish a review and coördination of high school work and provide a better background and introduction to university work. We have watched with interest the intelligent student expecting soon to leave school; he is quite as keen about broadening his intellectual horizon as the prospective university freshman who needs to be fortified as well as possible for his new job. The voluntary expressions of appreciation of former graduates who have taken elective work is very gratifying and comes both from the university student and from young men in business.

**Redistribution of Material.** After the selection of appropriate material, the next problem in redistribution is its effective organization and control. The essential points to be considered are sequence, grouping, generalizing, and method of development. The orthodox, logical sequence is not the only one possible. With beginners we are more concerned to have self-consistent, interdependent *units* and a psychologically coherent whole than to reproduce a Euclidean chain or a philosophical masterpiece of logical perfection. It is important to group the material in *units* on the basis of a key proposition or a fundamental concept; for example, the angle-sum in a triangle, parallel lines and parallelograms, the Pythagorean group, measurement of areas and volumes, similar figures, circle measurement, and the like. Again, within the group, it is well to emphasize the relative importance of the theorems, giving reasons. It is a good plan for students to keep note-books and in review of a unit to outline and summarize. For example, they can summarize (1) ways of proving triangles congruent; (2) ways of proving lines equal; (3) ways of proving angles equal; (4) ways of proving a quadrilateral a parallelogram; (5) ways of proving lines parallel; (6) ways of proving triangles similar; (7) facts they have proved about parallel lines; (8) facts they have proved about regular polygons; (9) facts they have proved about two intersecting secants of a circle.

**Generalization.** Generalization is an extremely important phase of mathematical thinking—the term expresses the use of induction which seeks from a number of scattered details to discover a general statement or principle, thus involving the whole question of abstraction and of transfer. Guiding the efforts of children in generalization is one of the most interesting features of the teaching of geometry. Some illustrations of general statements which include a group of theorems are: A triangle is determined if any three independent elements are given (except in the ambiguous case); if two intersecting lines cut a circle, the product of the segments of one equals the product of the segments of the other, measuring from the point of intersection of the secants to the points where the secant meets the circle; the angle formed by two lines which meet a circle has the same measure in degrees as half the sum of the intercepted arcs, half their difference, or half the arc, according as the lines meet inside, outside, or on the circumference. The Pythagorean theorem has many possible forms of generalization; we have sometimes had classes in which each individual presented a different proof, as well as taking the supplementary project of proving several of the general theorems of which this is a particular case.

**The Function of the Teacher.** The function of the teacher in helping the learner to learn is quite as important in a geometry class as an appropriate and well organized body of subject matter. Clear, forceful, and conclusive thinking can be acquired only by long and painful effort—this in part accounts for the fact that very few persons ever become thinkers. Indeed, I think teachers are often greater obstacles to progress than dull children or difficult work. At one time I could not understand why such a vigorous reform movement as Perry initiated should require a generation of thirty years to become an active agent in the ordinary classroom; however, after ten years' experience in teacher training, associated with the University of California, I have become deeply impressed with the persistence of tradition. It takes a long time to develop in a young university graduate the problem-solving attitude in his work as a teacher and in his relation to the work of the learner. He is prone to look upon the teaching process as forceful feeding, disgorging, and giving examinations; as a rule, he expects to teach the same things he was taught and to teach them in the same way.

Unfortunately many of our candidates for positions as teachers of mathematics did not study solid geometry in high school and have found no suitable opportunity to take it in their university course. We must deal with university graduates having majors in mathematics who are not at home in the subject matter of high school geometry and it is somewhat difficult for them to coöperate in administering a correlated course. Our hope for improvement is that the glorious exceptions may be sufficiently brilliant and strong in leadership to dominate the situation in the course of time.

**The Logical-Formal Process.** While it is not the purpose of this chapter to discuss method, it might be legitimate to do so, as the *logical-formal process* is strictly the characteristic subject matter of demonstrative geometry. It seems best to pass over the question of method, merely giving some of the most helpful and significant references. These are given in the footnote below.[1] We are interested in developing an active and effective problem-solving kind of thinking in children; in developing the power to meet new situations and the ability to investigate and discover facts to prove what they believe to be true, and to convince others of the truths they have discovered. This logical-formal material becomes in a sense a large part of the subject matter of the course, but the important thing to recognize is that such abstractions do not make wholesome food for babes. The difficult problem of reorganization is to discover exactly what type of concrete material does appeal to the interest of children and challenge their efforts. I believe that we are seeking the psychological order. There must be no abrupt transition from introductory intuitive geometry to systematic demonstrative geometry. A good deal of unconscious mental play and random experimenting may be permitted in the early stages by introducing the novel, beginning with concrete situations to be met, having a genuine problem. Some of the current social problems may be discussed with the students. Some day they should be qualified to coöperate in solving problems in their own social group. How did the Institute of Pacific Relations attack their problem? They defined their job—an experiment in understanding; their object was to study the conditions of the Pacific peoples with a view to improving their mutual relations; they then proceeded to develop a technique for their job. The method of attacking geometry problems should be applicable to problems of mechanical and social engineering as well—to problems of world peace, farm relief, temperance, tariff, or making a living.

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[1] Book, Wm. F.: *Learning How to Study and Work Effectively*, Chap. XX.
Columbia Associates: *An Introduction to Reflective Thinking.*
Colvin, S. S.: *The Learning Process.*
Dewey, John: *How We Think.*
Freeman, F. N.: *Psychology of the Common Branches*, p. 216.
Hassler, J. O.: "Analytic and Inductive Method vs. Synthetic and Deductive in Teaching Problem Solution." *School Science and Mathematics*, March 1925.
Hassler, J. O.: "Teaching Geometry into Its Rightful Place." *The Mathematics Teacher*, October 1929.
Johnson, Elsie Parker: "Teaching Pupils a Conscious Use of a Technique of Thinking." *The Mathematics Teacher*, April 1924.
Keyser, Cassius: *The Human Worth of Rigorous Thinking.*
Keyser, Cassius: *Thinking About Thinking.*
McLaughlin, H. P.: "Geometry as a Course in Reasoning." *The Mathematics Teacher*, December 1923.
Overstreet, H. A.: *About Ourselves.*
Shaw, James Byrnie: *The Philosophy of Mathematics.* Chap. V, "The Methods of Mathematics."
Shaw, James Byrnie: "The Way Mathematicians Work." *The Mathematics Teacher*, March 1923.

**Summary.** Summarizing our findings from experimenting with the redistribution of geometry material: it is the ever-enduring process of trying to adapt procedure to what we think is going on in the minds of the children we are trying to teach, and to the aims we have together set up as a goal. The teachers[2] who have coöperated in the experiment are unanimous in recommending a more fundamental basis for organizing elementary and advanced geometry than a division into plane and solid geometry; an increased number of postulates and a decreased number of theorems to be proved in the elementary tenth year course; close correlation of plane and solid geometry, trigonometry, algebra, and arithmetic; significant originals and applied problems; organization of subject matter into a coherent whole made up of self-consistent interdependent units. I have attempted to explain some of the working details of the general plan. The experiment has the glories of the incomplete. No textbook fulfills the requirements of the outline described above, although we have been most grateful for the suggestive and illuminating material in certain recent texts, especially with reference to instructional tests, illustrative material, live, up-to-date problems, and interesting challenges to student effort. Further adjustments in content may be made as well as improvement in the technique of teaching. We can, however, furnish statistics from the Oakland Department of Research and Guidance and from the records of our high school graduates to defend adequately our theories and practice.

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[2] The classroom teachers and supervising teachers of our University High School belong to the regular teaching staff of the Oakland Public Schools. The members of the Mathematics Department have been: Miss Anna Grafelman, Miss Nina Hospe, Miss Kate Foster, Miss Irene Lorimer, Miss Emma Hesse, Mr. Max Yulich, and the late Miss Ethel Durst. As head of the department I would like to express my appreciation of their excellent work and their fine coöperation. It is invaluable to have teachers who are in truth thinkers and teachers of thinking.

# A NEW APPROACH TO ELEMENTARY GEOMETRY

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**Purpose of Demonstrative Geometry.** In demonstrative geometry the emphasis is on reasoning. This is all the more important because it deepens geometric insight. To the extent that the subject fails to develop the power to reason and to yield an appreciation of scientific method in reasoning, its fundamental value for purposes of instruction is lessened.

There are, to be sure, many geometric facts of importance quite apart from the logical structure. The bulk of these belong properly in the intuitive geometry of Grades VII and VIII, and are not the chief end of our instruction in demonstrative geometry in the senior high school.

If, then, we find that some of our pupils in demonstrative geometry make little if any gain in the power to reason as a result of our instruction, and find, moreover, that other pupils of equal intelligence and schooling—except that they have had no demonstrative geometry—show very marked improvement in their ability to reason, our position is certainly open to attack. We cannot blame the pupils entirely. Perhaps there is something wrong with the subject; or perhaps the teaching could be improved.

**Undefined Terms.** If we are to give our pupils an appreciation of scientific method in reasoning, we ought to insist on a few undefined terms at the outset. Why pretend to define everything? The words "straight line" mean more to most people than Euclid's definition, or later paraphrases of it. Then why not call "straight line" an undefined term? If we cannot define "point," "surface," "angle," without involving the concept we are defining, why not take them as undefined?

**Need of Certain Assumptions.** Do our students appreciate the need of certain assumptions as fundamental in our logical system? Do we not allow them to infer that in geometry we can prove everything, and that we assume certain elementary propositions not because we have to, but because we are in a hurry to get on to more important matters? Should we not rather seize the opportunity to impress on them the need of certain assumptions, and show that this need is not peculiar to geometry, but is inherent in all logical systems? Could anything but good come from our indicating the possibility of some latitude in the choice of assumptions for geometry, and showing that each such choice leads to a slightly different approach to geometry, each valid with respect to its own group of assumptions and to no other? It is neither necessary nor desirable, perhaps, to mention non-Euclidean geometries in this connection.

What is the point in telling beginners that we shall assume certain "self-evident truths," and then asking them to *prove* certain other propositions which they regard as equally self-evident truths? Would they not come to a quicker understanding of the nature of a proof through the effort to prove easy "originals" which are not too plausible and which seem therefore to require justification?

**The Method of Proof by Superposition.** If we can possibly avoid it, should we continue to demoralize our classes at the outset by asking them to prove the obvious by the method of superposition, a method so out of harmony with the larger aim of our instruction that even though we recognize its validity we restrict its use to those few cases for which we can find no better method?

**The Incommensurable Case.** Euclid developed the arithmetic and algebra he needed by purely geometric methods, and made no reference to number. Confronted by "the incommensurable case," he was able to circumvent it only by means of inequalities and an exceedingly shrewd definition of proportion. Our instruction in arithmetic and algebra employs concepts developed since Euclid's time and takes number as its starting point. Our pupils are not troubled by irrationals and see no need of the incommensurable case. Our present practice is to pay but scant attention to it; but that hardly removes the difficulty, because it still stands in print before us. With more effective use of our irrationals we could get along without any mention of incommensurable cases; but for Euclid they were of fundamental importance. If we mention the incommensurable case at all, we can hardly dismiss it as exceptional, for the incommensurable is of relatively common occurrence: the commensurable is the exception.

**Texts in Geometry.** In short, many pupils get little good from their study of geometry because at certain critical points the text is quite inadequate. This is of course no disparagement of Euclid: his audience knew more about logic and less about number than do our pupils in school to-day. From our point of view his text should say much more about the foundations of logical method, connecting these with situations outside geometry by appropriate exercises, and should start with the real number system as we have it to-day and gain the power and simplicity which such an approach can yield.

**Possible Changes in Teaching Geometry.** Perhaps if we are to rewrite Euclid we should consider other questions too. Congruence and parallelism are fundamental in Euclid's geometry; and from parallelism is derived the concept of similarity. We can express this symbolically as follows: $=, \parallel \rightarrow \sim$. But in our demonstrations we refer chiefly to similarity, and less often to those aspects of parallelism not comprised in similarity. Moreover, congruence and similarity have much in common. The British Report on *The Teaching of Geometry in Schools* (G. Bell and Sons, London, 1923, p. 35) makes the interesting suggestion that we replace the Parallel Postulate by a Postulate of Similarity, and derive the idea of parallels from similarity. This arrangement can be shown symbolically as follows: $=, \sim \rightarrow \parallel$.

Whether we make such a change or not, our pupils should see that the Parallel Postulate can be replaced by some other assumption—for example, a proposition concerning equal corresponding angles formed by a transversal and two other lines—and that the erstwhile Parallel Postulate is now a theorem depending on the new assumption. The pupils should also discover the effect on geometry of omitting the Parallel Postulate, or its equivalent, and all the theorems dependent on it.

**Plane and Solid Geometry Combined.** Inasmuch as most students of demonstrative geometry devote but one year to the subject, it will probably be worth while to make as many allusions as possible to related propositions in three dimensions. This three-dimensional material would have to be based on intuition and find expression mainly in the exercises. The principle of duality and certain other modern concepts should be included also.

**An Approach Based on Number.** Let us give further consideration to an approach to geometry based directly on number.

Within the last century Riemann devised an approach to geometry, not intended for elementary purposes and quite unsuited to them, but nevertheless very suggestive. In this geometry the notion of *distance* between two points is regarded as the primary geometric relation. A little thought will make it plain that most geometric conceptions can readily be defined in terms of distance. For example, the line segment $AB$ may be defined as the collection of points $P$ such that the distance from $A$ to $P$ plus the distance from $P$ to $B$ is equal to the distance from $A$ to $B$. Again, the circle may be defined as the plane figure consisting of all the points $P$ at a given distance $r$ from a fixed point $O$ called the center. From the Riemannian point of view geometry then appears as the theory of the interrelation of various distances between points, and the formula which expresses the distance between any two points is regarded as the basic element.

In the case of Euclidean geometry this formula may be written

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

where the pairs of numbers $x$ and $y$ are the labels which identify the point in question. This asserts that the distance between any two points $P_1$ and $P_2$ whose number pairs are $x_1$, $y_1$, and $x_2$, $y_2$ is given by the square root of the sum of the squares of the differences of the $x$'s and $y$'s corresponding to these two points. Here we are not to think of the formula in itself as having geometric significance; instead we must think of a vast collection of points, the distance of any two of them being given to us by tabular entries, for instance, and then we must think of the above formula as giving a particularly simple rule by which the various points could be identified and the distances between them found as given in the fundamental table by short numerical reckoning.

**Advantages of Such a Method.** Now with this Riemannian method of approach intuitive processes which are fundamental for Euclidean geometry have no place. That indeed is the fundamental difficulty with the method from a pedagogical point of view. But from a logical point of view it has several advantages which ought to be pointed out. In the first place, whereas Euclidean geometry takes for granted that there exist such things as points, lines, and planes, although as a matter of fact none actually correspond to physical objects, in this new method the whole construction is based upon the number system. For example, a point is defined to be a pair of numbers $x$, $y$ and the distance between any two points is defined as in the above rule. Consequently we know at the outset that we are dealing with entities which exist in the same sense that numbers do, so that our conclusions about them are bound to be as consistent as our rules of reckoning with numbers. In the second place, when we employ this method of approach we build up at the same time the elementary ideas which belong to analytic geometry and thus introduce the student not only to the ordinary geometric theorems but also to their formulation in terms of concepts of analytic geometry.

**Illustrative Examples.** A simple illustration will bring out both of these points. Suppose that we are seeking the points on the line segments which join point $A$ $(0, 0)$ to the point $B$ $(1, 1)$. If $P$ $(x, y)$ represents any point of this segment, then, by the definition above, $AP + PB = AB$ and

$$\sqrt{x^2 + y^2} + \sqrt{(1 - x)^2 + (1 - y)^2} = \sqrt{2}.$$

If now we transpose one of the terms on the left, square and simplify, we find by a little easy algebra that

$$x + y = \sqrt{2}\,\sqrt{x^2 + y^2}.$$

If both members of this equation are squared again we obtain the equation

$$x = y,$$

which defines the line upon which all points of the segment must lie. It is then easy to show that only those points are to be taken for which $x$ $(= y)$ is numerically less than 1 and positive. Here, then, we have illustrated how the natural definition in terms of distance leads to the formulation of the equation of a line in a simple case. From this point of view questions concerning intersecting lines and non-intersecting or parallel lines reduce themselves to algebraic questions as to whether or not certain pairs of linear equations in two unknowns $x$ and $y$ have or have not a solution.

**The Pythagorean Theorem.** As another illustration we might refer to the Pythagorean theorem. How does the Pythagorean theorem appear from the Riemannian point of view? In the first place, we may regard the elementary formula itself as a formulation of the Pythagorean theorem, at least when the triangle in question has two of its sides parallel to the $x$ and $y$ axes. But more generally, we should first have to define two perpendicular lines. To this end we could define the perpendicular $PM$ dropped from $P$ to the line $l$ and meeting $l$ at the point $M$ as that line for which the distance $PM$ is as small as possible. By purely algebraic manipulation it turns out that given any two line segments $AC$ and $CB$ such that $AC$ and $CB$ are perpendicular, then $(AB)^2 = (AC)^2 + (CB)^2$.

**The Euclidean Method.** Considering, then, these two contrasting methods of approach to elementary geometry, we may say that the Euclidean method proceeds from qualitative propositions called postulates, not involving number at all (to which we therefore attach the symbol 0), to other propositions involving linear and angular measurement (to which we accordingly attach the number 1), and finally to propositions which involve number in a two-dimensional way (to which we therefore attach the number 2). Of this last class the Pythagorean proposition and the theorem that the sum of the three angles of a triangle is 180° may be regarded as the typical and most important instances. Thus the processes of development in the Euclidean approach may be indicated schematically by the following diagram:

$$0 \rightarrow 1 \rightarrow 2.$$

**The Riemannian Approach.** In the Riemannian approach we start with a formula which involves number in a two-dimensional way at the outset, for the fundamental formula really embodies the Pythagorean theorem. Then from this formula, by means of suitable definitions and the use of algebraic methods, we deduce other propositions, such as those dealing with linear and angular measure and also the qualitative propositions with which we started in the Euclidean case. For example, the proposition that two points determine a straight line would involve first the definition of the straight line as indicated above, second the proof that any equation of the first degree in $x$ and $y$ represents a straight line, and finally that one and essentially only one equation can be found which is satisfied by two given distinct pairs of numbers $x_1$, $y_1$ and $x_2$, $y_2$. In consequence the diagram which we use to characterize this method of approach is the following:

$$2 \rightarrow 1 \rightarrow 0.$$

**Disadvantages of Each Method.** The disadvantages of both methods are obvious. Euclid's method is circuitous and does not take advantage of well known facts concerning number and linear and angular measurement in terms of number. These facts, which appear absolutely self-evident at an early age under present conditions of training, must in this scheme be regarded as things to be demonstrated, at least if a purely logical point of view is adopted. The method of Riemann, on the other hand, is totally devoid of intuitive significance and involves fairly difficult algebraic manipulations at the outset.

Each method has, however, its advantages, which have been referred to above.

**A New Approach to Elementary Geometry.** With these preliminaries let us attempt to formulate a method of approach which may possibly eliminate most of these disadvantages and at the same time embody the fundamental advantages of both methods. This method may in brief be characterized by this diagram:

$$0 \leftarrow 1 \rightarrow 2.$$

In this case we take for granted at the outset the notion of number and assume that the student is capable of making simple computations by means of number. Also we admit the self-evident fact of linear and angular mensuration and scale drawing; that is to say, we accept the simple facts familiar to any boy or girl who knows how to use ruler and protractor. On the basis, then, of four or five simple postulates of this type, the most important geometric conclusions which are not self-evident can be rapidly developed. Among these would be the Pythagorean proposition and the theorem that the sum of the angles of a triangle is 180°. Furthermore, on the basis of these postulates and these fundamental theorems all other theorems in geometry can be derived easily and naturally.

**Fundamental Principles.** The fundamental principles necessary to such a development may be taken as follows:

I. The Principle of Line Measure.—*The points on any straight line can be numbered so that number differences measure distances.*

II. *There can be only one straight line through two given points.*

III. The Principle of Angle Measure.—*All half lines having the same endpoint can be numbered so that number differences measure angles.*

IV. *All straight angles have the same measure, 180°.*

V. The Principle of Similarity[1] (Part I).—*Two triangles are similar, if an angle of one equals an angle of the other and the sides including these angles are proportional.*

**Basic Theorems.** By means of these five assumptions we can prove the following six basic theorems:

VI. The Principle of Similarity (Part II).—*Two triangles are similar if two angles of one are equal to two angles of the other.*

VII. *If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.*

VIII. The Principle of Similarity (Part III).—Two triangles are *similar if their sides are respectively proportional.*

IX. *The sum of the three angles of a triangle is 180°.*

X. *Through a point on a line there exists one and only one perpendicular to the line.*[2]

XI. The Pythagorean Theorem.—*In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides; and conversely.*

**Corollaries.** As corollaries of XI we have the following:

XIa. *The altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.*

XIb. *If the hypotenuse and side of two right triangles are in proportion, the two triangles are similar.*

XIc. *The sum of two sides of a triangle is greater than the third side.*

XId. *The shortest distance from a point to a line is measured along the perpendicular from the point to the line.*

XIe. *Of two oblique lines drawn from a point to a line, the more remote is the greater; and conversely.*

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[1] Principle V is very powerful. Though stated in terms of similarity and proportion, it evidently includes the case of two congruent triangles with two sides and included angle respectively equal. It is apparent from the content of these five fundamental principles what terms we must define, or take as undefined.

[2] Theorem X should be taken for granted at the outset; for although it can be derived from the basic principles, most beginners would hardly care to question it. At the end of the course they can return to a consideration of the basic principles underlying their geometry and can prove this proposition which formerly they took for granted. This same procedure should be applied to one or two other theorems whose content seems obvious to the beginner. He should make a list of all the propositions he takes for granted and compare it at the end of the course with the minimum list on which the geometry is based.

**Treatment of Parallels.** The treatment of parallels can be derived from the principle of similarity. By means of this principle also we can show that any two perpendicular lines (coördinate axes), together with all the lines at right angles to them, form a rectangular network; and that all the lines perpendicular to an axis are parallel.[3] The concept "slope of a line with respect to a network" follows also from the *Principle of Similarity* and leads at once to the equation of the straight line.

**Area of a Triangle.** The area of a triangle is a number, constant for the triangle, and equal to $k$ times the product of a side and the altitude upon it. We may assign any value we like to $k$; so we choose such a value as will make the area of the unit square equal 1. This means that $k$ must equal $\frac{1}{2}$.

**Advantage of Pythagorean Theorem.** It is a tremendous advantage to use the Pythagorean theorem from the very beginning, especially in connection with early propositions concerning the circle.

**Possible Constructions.** It is interesting to see what constructions were possible for Euclid with only unmarked straight-edge and compasses; but it is comforting also to have scale and protractor always at hand and to know that it is good form to use them.

**Advantages of the New Approach.** Let us see how this program stands in comparison with the Euclidean and Riemannian programs, from the standpoint not only of mathematical importance but of pedagogical usefulness. In the first place, it is severely logical. In this respect it offers as satisfactory training as that of Euclid, and is much more direct. It lends itself admirably to explicit consideration of the place of undefined terms, definitions, and assumptions in any chain of logical reasoning and leads to the development of those same habits, attitudes, and appreciations which all teachers of geometry claim for their subject. In the second place, it takes advantage of the knowledge of number and of linear and angular mensuration which the student possesses, and so does away with the feeling of artificiality which is inevitable when seemingly self-evident propositions are "proved." In the third place, it leads very naturally to the elementary facts of analytic geometry and makes it apparent in this way that geometry is really a self-consistent discipline whether or not such things as points, lines, and planes really exist.

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[3] See Birkhoff, George D., *The Origin, Nature, and Influence of Relativity*, p. 36. Macmillan Co., 1925.

**Value of Study of Geometry.** It is true that some of our pupils seem to derive little or no profit from their study of geometry. Perhaps some teachers take comfort in the notion that their subject possesses a disciplinary something capable of "transfer" to situations outside geometry and forget that they must do their share to encourage the transfer. The text usually fails in the same respect and has other shortcomings, as we have been at pains to point out. But the remedy for these ills is not the present popular mode, complete abolition of the subject in question, or an almost equally dire emasculation. For although it is difficult to prove that the study of geometry necessarily leads in large measure to those habits, attitudes, and appreciations which its advocates so eagerly claim for it, it is even harder to prove that under proper conditions it cannot be made to yield these outcomes, and more readily than other subjects of instruction. Should we then abolish geometry from the secondary school, or should we try first to reform it?

It is often said that the Euclidean approach to elementary geometry was designed for thoroughgoing scholars, and was not intended for the immature youth of to-day. That is of course true; but it should not be taken to mean that geometry is beyond the ken of pupils in secondary schools simply because it was not written with them in mind. Youngsters in the grades to-day grapple with arithmetic and algebraic intricacies, many of which were subjects of debate among adult Greeks of Euclid's time, and some of which it was reserved for adults of relatively recent times to discover. Later discoveries often shed light on earlier revelation and render easy what once was hard. That has been true of algebra, and could be equally true of geometry if we but dared to take the step.

We have suggested a modification of Euclid in accord with psychological and mathematical ideas which, though commonplaces to us to-day, were not available to Euclid. This modification would at once simplify the subject and give it greater significance.

In an age when the amount of material of scientific importance which the student ought to learn has become very large and the demands upon his time are numerous, it would seem that the possibilities of this new method of approach should be thoroughly investigated.

# GRAPHIC METHODS OF TEACHING CONGRUENCE IN GEOMETRY

By JOHN A. SWENSON

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**Influence of Euclid.** The sway which Euclid's *Elements* has held as a textbook for more than two thousand years is without parallel in the history of mathematics. Even the invention of Cartesian geometry in 1637 has not affected the teaching of the so-called Euclidean geometry. An almost unlimited number of textbooks have appeared in modern times but the only way in which they have differed is in the sequence of the theorems. Euclid's treatment has in the main been retained and no modern mathematical methods have been introduced.

**The Method of Proof by Superposition.** The Report of the National Committee on Mathematical Requirements emphasizes the function concept but there is nothing in the report to indicate that even this ultra-modern committee had in mind anything essentially different from the usual Euclidean treatment. This treatment may be best suited to the beginner's requirements and the one most readily comprehended by him.

We can never be absolutely sure of this, however, until we have experimented with other methods. Other considerations also enter. The mathematician does not consider the method of proof by superposition very satisfactory. The psychologist says we should introduce no unnecessary habits in connection with the processes of learning. The use of superposition is surely one of these unnecessary habits because it is barely introduced before it is discarded.

**Teaching Congruence by Graphic Methods.** In connection with experiments to make variation the fundamental tool in both algebra and geometry, I have substituted graphic methods for superposition. After the first congruence theorem has been verified experimentally by the actual cutting out and placing of one triangle on the other, the other congruence theorems are treated graphically in the manner indicated below.

**Variation in Triangle *ABC*.** In Fig. 1a, *ABC* is a triangle in which $AB = 2$ in.,* $\angle BAC = 30°$, and $\angle ABC = 15°$. If we keep *AB* and $\angle BAC$ fixed or constant (i.e., retain the values 2 and 30°, respectively) and allow $\angle ABC$ to increase, say, by steps of 20° so that $\angle ABD = 35°$, $\angle ABE = 55°$, and so on, then the opposite side *AC* will also increase and become *AD*, *AE*, *AF*, and so on, as shown in the accompanying table.

| Triangle<br>△ | Angle<br>$x$ | Side<br>$y$ |
|---|---|---|
| *ABC* | 15° | *AC* |
| *ABD* | 35° | *AD* |
| *ABE* | 55° | *AE* |
| *ABF* | 75° | *AF* |
| . | . | . |
| . | . | . |
| *ABA* | 0° | *AA* |

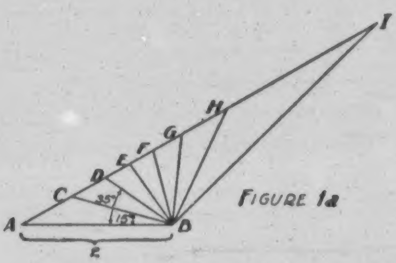

FIGURE 1a

To show how these angles and the opposite side vary together throughout the whole range of possible values, let us construct a graph, using $\angle$s *ABC*, *ABD*, *ABE*, *ABF*, and the like, as $x$'s and the opposite sides *AC*, *AD*, *AE*, *AF*, and the like, as $y$'s. (See Fig. 1b.)

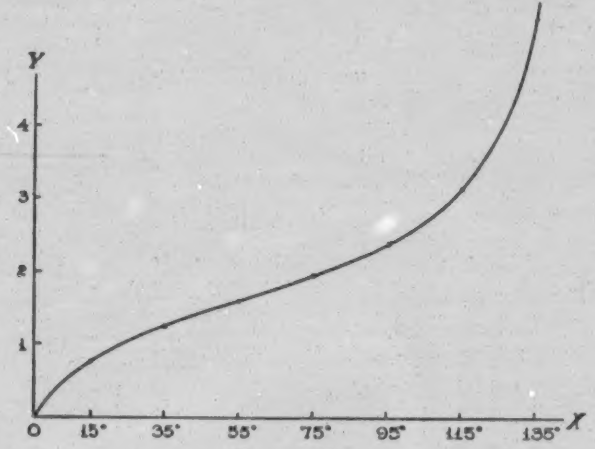

FIGURE 1b

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* The figures in this chapter have been reduced in size from the original.

In constructing the graph, the $y$'s are taken by direct measurement from the triangles, but the $x$'s are laid off directly by letting one linear unit represent 5°. An inspection of the graph shews clearly:

1. If we traverse the graph from left to right, $x$ and $y$ increase together.
2. If we move from right to left, $x$ and $y$ decrease together.
3. To each value of one variable corresponds only one value of the other.

When these three conditions are fulfilled, we say that $x$ and $y$ vary in the same sense throughout the whole range of possible values.

**Variation in the Opposite Sense.** Any two variable magnitudes may be used in constructing a graph. To obtain a graph of a different nature from the graph in the preceding figure, we shall again keep $AB$ and angle $A$ in Fig. 1a constant and use $BC$, $BD$, $BE$, $BF$, and so on, as $x$'s and $AC$, $AD$, $AE$, $AF$, and so on, as $y$'s, as indicated in the table below. The graph from these data shows plainly that as $y$ increases, $x$ decreases for a while and then begins to increase.

| Triangle $\triangle$ | Side $x$ | Side $y$ |
|---|---|---|
| $ABC$ | $BC$ | $AC$ |
| $ABD$ | $BD$ | $AD$ |
| $ABE$ | $BE$ | $AE$ |
| $ABF$ | $BF$ | $AF$ |
| . | . | . |
| . | . | . |

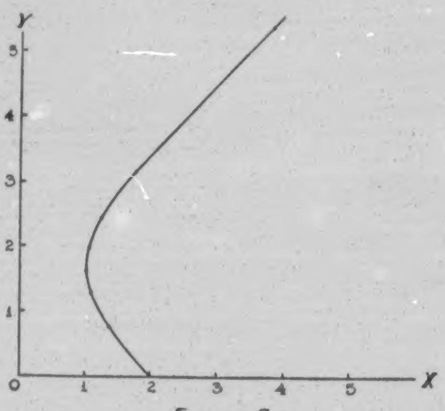


FIGURE 2

The graph in Fig. 1b shows no such fluctuation. In that graph $x$ and $y$ both vary in the same sense throughout the whole range of values. In the graph in Fig. 2, $x$ and $y$ vary in the opposite sense for a while and after that in the same sense, but there is no consistency throughout the whole graph such as we encountered in the first graph.

*Exercises.*

1. Using the preceding graphs, determine
   a. the values of $x$ which correspond to certain specific values of $y$.
   b. the values of $y$ which correspond to certain specific values of $x$.
   c. the change in $x$ which corresponds to certain specific changes in $y$.
2. Specify that part of the graph in Fig. 2 which shows that $x$ and $y$ vary (a) in the same sense; (b) in the opposite sense.

**Use of Graphs in Proving Congruence.** The preceding sections have shown the truth of the following principles:

If in a triangle one side and one of its adjacent angles remain constant, then

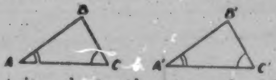
1. *the other adjacent angle and its opposite side vary in the same sense through the whole range of possible values.*
2. *the other two sides do not vary together in a consistent manner.*

By using the first principle above we can prove the following:

*Theorem. Two triangles are congruent if two angles and the included side of one triangle are equal respectively to two angles and the included side of the other.*

*Given* $\triangle ABC$ and $\triangle A'B'C'$ in which $AC = A'C'$, $\angle A = A'$, $\angle C = \angle C'$.

*Prove* $\triangle ABC = \triangle A'B'C'$.



*Proof.* In passing from the first triangle to the second, $AC$ and $\angle C$ remain constant ($AC = A'C'$, $\angle C = \angle C'$). Hence $AB$ and $\angle C$ vary in the same sense, which means that $AB$ must remain constant when $\angle C$ does. But it is given that $\angle C$ remains constant. Hence $AB$ remains constant, or $AB = A'B'$, and the two triangles are congruent by the first congruence theorem.

*Exercise.* Show that it is not possible to prove two triangles congruent when two sides and an angle opposite one of them in one triangle are equal respectively to the corresponding parts in the other.

**Another Variation in Triangle *ABC*.** In Fig. 3a, $ABC$ is a triangle in which $AB = 1.5$ in., $AC = 2$ in., and $\angle BAC = 25°$. If we now keep $AB$ and $AC$ constant (i.e., of the same length), and allow $\angle BAC$ to increase by steps of 20°, then we have the set of triangles described in the following table. The graph (Fig. 3b) shows that $x$ and $y$ vary in the same sense. Hence: *If two sides of a triangle remain constant, the included angle and the third side vary in the same sense.*

| Triangle | Angle | Opposite Side |
|---|---|---|
| | $x$ | $y$ |
| $ABC$ | 25° | $BC$ |
| $ABD$ | 45° | $BD$ |
| $ABE$ | 65° | $BE$ |
| $ABF$ | 85° | $BF$ |
| $ABS$ | 165° | $BS$ |
| $ABT$ | 180° | $BT$ |
| $ABM$ | 0° | $BM$ |

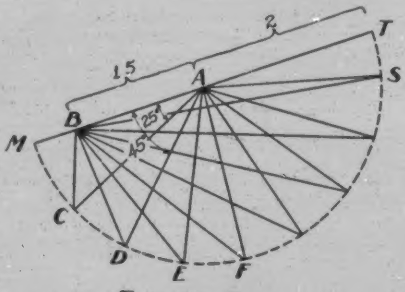


FIGURE 3a

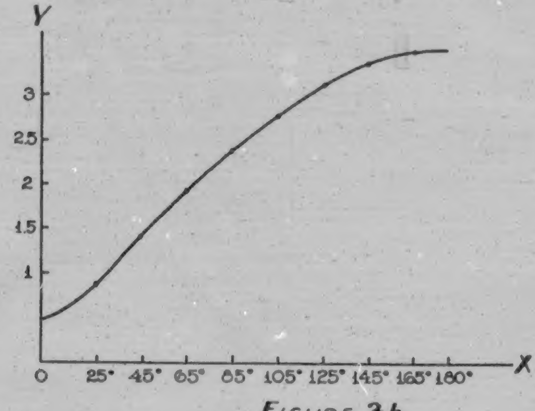


FIGURE 3b

From this theorem it follows readily that *two triangles are congruent if three sides of one triangle are respectively equal to three sides of the other.*

**Difference Between Algebra and Geometry.** Modern inventions have produced such powerful tools that it is hardly excusable to confine oneself exclusively to those in use 2,000 years ago. If Euclid lived to-day he would no doubt take advantage of modern algebraic methods. When the modern teacher of mathematics keeps algebra and geometry in separate compartments, he sacrifices much that a union of the two subjects can bring out.

**Summary.** The above sketch is a mere hint of how the two subjects may be used to reinforce each other in bringing out the fact that mathematics is fundamentally a study of variation. In nature and everyday life all things change, and it is the business of mathematics to study these changes and express the laws according to which they take place. The topic *Superposition* has been chosen purposely because it is generally argued that this is the topic in geometry that least of all fuses with algebra.

# THE USE OF INDIRECT PROOF IN GEOMETRY AND IN LIFE*

By CLIFFORD BREWSTER UPTON

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**Indirect Reasoning in Life.** Recently a valve on the steam radiator in my office gave evidence of being defective; when the valve was turned off, steam continued to flow into the radiator. I called the college engineer who removed the valve, cleaned it, and then replaced it. This procedure, however, failed to remedy the trouble for no matter how long the valve was closed, the radiator remained hot. I called the engineer again and told him that the valve was defective and that a new valve was needed. He contended that the valve was a good one and that a new one was unnecessary. That explanation didn't satisfy me since the radiator was still hot though the valve had been tightly closed all night long. I then asked him, "If that were a good valve, could steam leak through the valve when it is turned off?" He admitted that steam could not leak through it if the valve were a good one; hence he too was convinced that the valve was defective. The valve was removed and replaced by a new one, and the trouble was remedied.

This is a good example of a type of reasoning that occurs rather frequently in life. It is a natural kind of argument, it comes spontaneously, and in general it is convincing. We call it *indirect* reasoning because we did not prove *directly* that the valve was defective. We established this fact *indirectly* by showing that it was *not* a good valve.

This indirect method of proof, which is often called the method of *reductio ad absurdum* (reduction to an absurdity) is also used in our high school work in demonstrative geometry. In fact, a few of our basic theorems, and certain exercises, can be proved only by the *indirect* method.

**Some Criticisms of Indirect Proof.** It is of interest, however, that though indirect proof is easily understood when applied

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in many life situations, it is, on the contrary, considered by a large group of teachers as more or less unsatisfactory when used in geometry. Just why it is unsatisfactory is not easy to explain, yet the fact remains that many teachers find *indirect* proof much less convincing than *direct* proof. As evidence of this dissatisfaction, I quote below the statements of a number of geometry teachers, recently attending the Summer Session courses at Teachers College, who were asked to give their impressions concerning indirect proof:

1. Pupils feel that indirect proof "beats around the bush."
2. Indirect proof is not always convincing. It often seems absurd to pupils.
3. The theorems proved by it are often too nearly self-evident.
4. Indirect proof tends to produce reasoning in a circle because of the informal way in which it is used.
5. If the theorem is true, why can't a direct proof for it be found?
6. The indirect method is hard to teach; it is not a natural method.
7. The use of the indirect method often results in memorizing the proof.
8. Teachers do not really understand it.
9. It is better to prove positively than to disprove negatively.
10. Indirect proof is usually taught too early when pupils have many other things to learn.
11. The ability to disprove each of the false assumptions is beyond most of the pupils.
12. Pupils do not see all the possibilities; they draw conclusions too soon.
13. Pupils do not seem to know when the proof is finished.
14. Pupils get the idea that we are dodging the issue.
15. The figure used to represent the false assumption cannot be accurately drawn.
16. Pupils do not know when to apply indirect proof.
17. It is not really a proof.

This feeling that indirect proof is not as satisfying as direct proof is not limited to high school teachers. Coffey, in his *Science of Logic*, states that "indirect proof is obviously less satisfactory and less scientific than direct proof for it does not give the mind any insight into the positive intrinsic causes or reasons why the established proposition is really true. Nevertheless, it is of great importance as a path to certain knowledge and it is used extensively in every department of research." [1] The famous mathematician Augustus De Morgan, who wrote texts on formal logic as well as on mathematics, states that "indirect proof is as logical as direct proof but not of so simple a kind, hence, it is desirable to use direct proof whenever it can be obtained."[2] Dauzat, the well-known French writer on methodology, states that "indirect proof is a sure method, but although it is convincing, it is not illuminating, and should be used only as a last resort."[3]

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[1] Coffey, *The Science of Logic*, Vol. II, p. 233.

**Importance of Indirect Proof.** It should be pointed out that although the above comments indicate that indirect proof is not as satisfying as direct proof, yet all logicians recognize the importance and logical soundness of the indirect method and certain of them emphasize our great dependence upon it in everyday life. For example, Jevons, one of the great authorities on logic, goes so far as to say that "nearly half of our logical conclusions rest upon its employment."[4] Milnes, in his revision of De Morgan's *First Notions of Logic*, which was intended by De Morgan as an introduction to geometrical reasoning, states that "the process of *reductio ad absurdum* is of the greatest importance. It is the most prominent of all the methods by which men learn those truths of Nature that are unitedly known by the name of Science."[5]

In our high school work in demonstrative geometry it is impossible to develop a syllabus of propositions all of which shall be proved by direct methods. To establish certain propositions, we are obliged to resort to the method of *reductio ad absurdum*.[6] In Euclid's *Elements*, the first great textbook on geometry written about 300 B.C., we find in Book I that the indirect method is used to prove eleven basic propositions.[7] Our modern American writers on demonstrative geometry tend to limit the use of the indirect method as much as possible, though practically all of them use this method to prove at least five regular propositions or corollaries in Book I of their textbooks on geometry.

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[2] De Morgan, *Formal Logic*, p. 24.

[3] Dauzat, *Eléments de Méthodologie Mathématique*, p. 10.

[4] The entire paragraph in which the above statement appears is here quoted:

"Some philosophers, especially those of France, have held that the Indirect Method of Proof has a certain inferiority to the direct method, which should prevent our using it except when obliged. *But there are many truths which we can prove only indirectly.* We can prove that a number is a prime only by the purely indirect method of showing that it is not any of the numbers which have divisors. We can prove that the side and diagonal of a square are incommensurable, but only in the negative or indirect manner, by showing that the contrary supposition inevitably leads to contradiction. Many other demonstrations in various branches of the mathematical sciences proceed upon a like method. Now, if there is only one important truth which must be, and can only be, proved indirectly, we may say that the process is a necessary and sufficient one, and the question of its comparative excellence or usefulness is not worth discussion. As a matter of fact I believe that nearly half our logical conclusions rest upon its employment."—From Jevons, *The Principles of Science*, p. 82.

[5] Milnes, *Elementary Notions of Logic*, p. 93.

[6] Throughout this discussion the terms "indirect proof," "indirect method of proof," and "reductio ad absurdum" are used synonomously.

[7] Euclid was not the first one to use indirect proof in geometry. Eudoxus, who lived about 370 B.C., is known to have used this method of reasoning to prove certain theorems. See Allman, *Greek Geometry from Thales to Euclid*, p. 186.

Indirect proof is a very powerful instrument. To use it successfully, one must be thoroughly acquainted with it and have much practice in its use. The fact that the pupil encounters it so seldom in geometry and hence has so little practice in using it, probably accounts in large measure for much of the dissatisfaction expressed concerning it. In our geometry classes to-day, we spend considerable time explaining the nature of *direct proof;* we also give considerable attention and practice in the *method of analysis* as a means of discovering direct proof. Unfortunately, no corresponding amount of attention has been given to the *indirect method of proof* and it is this neglect, in all probability, that is the cause of the trouble. Another element contributing to this difficulty is the fact that the indirect method is most frequently used to prove the converses of certain propositions; it is the experience of most teachers that the notion of a "converse proposition" is, in itself, a source of more or less confusion to many pupils.

**Nature of Indirect Proof.** Recognizing the fact that the indirect method is not entirely satisfying to many of our high school teachers and pupils, and also accepting the fact that we are absolutely dependent upon this method, not only to establish a connected chain of propositions in geometry, but also in many life situations, it seems important for us to examine carefully the nature of indirect proof and to state its underlying principles from the standpoint of the science of logic. This discussion is intended for teachers with the hope of making available in a single article certain materials that would otherwise be obtainable only by consulting a wide range of literature on logic and on geometry; in many libraries certain of the books quoted would not be found.

Perhaps the simplest way to illustrate the principles involved in indirect proof will be to return to our illustration of the defective steam valve, which was presented at the beginning of this article. In that illustration we proved indirectly that the steam valve was defective by showing that it was *not* a good valve. In this reasoning, we assumed that one of two things *must be true,* either (*a*) the valve is a *good valve*, or (*b*) the valve is a *defective valve.* These statements (*a*) and (*b*) are such that "both cannot be true at the same time; likewise both cannot be false at the same time. In other words, at the same time *only one* of these statements can be true and *only one* can be false. If (*a*) is true, then (*b*) must be false, and if (*a*) is false, then (*b*) must be true; likewise, if (*b*) is true, then (*a*) must be false, and if (*b*) is false, then (*a*) must be true. There is no middle ground; there is no third possibility."[8] If we wish to prove that (*b*) is true, it suffices, therefore, to prove that (*a*) is false; such proof is called *indirect*. On the other hand, if we had established the truth of (*b*) without any reference to (*a*), which might *possibly* have been done by removing the valve and discovering that one of its inner parts was broken, then our conclusion would have been reached by *direct* proof.

Indirect reasoning of the kind just illustrated is also known as the method of *reductio ad absurdum*, because it always leads to an absurd conclusion. For example, in the illustration of the defective steam valve, the assumption that the valve was a good one led us to conclude that it would shut off the steam when tightly closed, but this was an absurd conclusion because the steam continued to flow when the valve was closed. Since to assume the valve good led to an absurdity, we concluded that our assumption was false. Hence the valve must have been defective since that was the only other possibility.

**Contradictory Propositions.** In the above discussion, we see the statement that *the valve is good* contradicts the statement that *the valve is defective;* such statements, in the science of logic, are called *contradictory propositions,* because one of them *contradicts* the conclusion of the other.[9] Contradictory propositions may be defined as follows: "Two propositions are contradictory when they are exact opposites; one must be true and the other must be false."[10]

Contradictory propositions play such an important part in the study of logic that we will now summarize certain important facts concerning them which have already been brought out:

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[8] Jones, *Logic, Inductive and Deductive,* p. 115.

[9] Other examples of contradictory propositions are as follows: Angle $A$ equals angle $B$; angle $A$ does not equal angle $B$. Lines $m$ and $n$ are parallel; lines $m$ and $n$ are not parallel. It is raining; it is not raining.

In connection with the contradictory propositions *it is raining* and *it is not raining,* one may at first hastily decide that these two propositions do not cover all the possibilities since it may be snowing or it may be hailing; it will be seen, however, that *it is snowing* and *it is hailing* both come under the proposition, *it is not raining.* Hence, the propositions *it is raining* and *it is not raining* represent the only two possibilities. These propositions are also such that when one of them is true, the other must be false, hence they are contradictory propositions in accordance with the definition given above.

[10] Jones, *Logic, Inductive and Deductive,* p. 115.

(1) *Two contradictory propositions cannot be true together.*

(2) *Two contradictory propositions cannot be false together.*

(3) *Of two contradictory propositions, one must be true and the other false.*

These facts may be summed up in the form of a working principle as follows:

*If one of two contradictory propositions is proved to be true, it immediately follows that the other must be false; similarly, if one of them is proved to be false, it immediately follows that the other must be true.*

This principle is applied when we prove a proposition by the method of *reductio ad absurdum,* since we prove the proposition to be true by showing that its contradictory proposition is false.

This discussion of contradictory propositions and their use in indirect reasoning is also closely related to two *fundamental laws of thought* which are described in most of the standard textbooks on logic. These laws are as follows:

(1) The Law of Contradiction, which states that *a thing cannot both be and not be.*

> "The Law of Contradiction points out that nothing can have at the same time and at the same place contradictory and inconsistent qualities. A piece of paper may be blackened in one part, while it is white in other parts; or it may be white at one time, and afterwards become black, but we cannot conceive that it should be both white and black at the same place and time. A door after being open may be shut, but it cannot at once be shut and open. No quality can both be present and absent at the same time; and this seems to be the most simple and general truth which we can assert of all things. Aristotle truly described this law as the first of all axioms—one of which we need not seek for any demonstration."[11]

(2) The Law of Excluded Middle, which states that *a thing must either be or not be.* This is also called the Law of Duality.

> "The Law of Excluded Middle asserts that at every step there are two possible alternatives—presence or absence, affirmation or negation. It asserts also that between presence or absence, existence and non-existence, affirmation and negation, there is no third alternative. As Aristotle said, there can be no mean between opposite

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[11] **From Jevons and Hill, *Elements of Logic,* p. 105.**

> assertions; we must either affirm or deny. Hence, the inconvenient name by which it has been known—The Law of Excluded Middle."[12]

**Another Important Principle.** It will now be helpful to state one other important principle upon which indirect reasoning is based. This can best be done by referring to our first illustration of the defective steam valve. When I called the engineer the second time, I wanted to convince him that the valve was defective. In other words, I wanted to prove the proposition: *The valve is defective.* To do this, I proved that the contradictory proposition, namely, that *the valve is good*, must be false. But the question arises, how did I prove this contradictory proposition to be false? I did so by proving that something which necessarily follows from it must be false. In the discussion with the engineer, the contradictory proposition, namely, that *the valve is good*, led me to the conclusion that the valve when tightly closed should shut off the steam. But I knew that this must be a false conclusion because the steam continued to flow when the valve was turned off. Hence, I concluded that the contradictory proposition, namely, that *the valve is good*, must be false, since it led me to a false conclusion.

Whenever the final conclusion of any piece of reasoning is known to be false, then one of two things must have happened, either we have made some error in our process of reasoning or we started from a false assumption. On arriving, therefore, at a conclusion known to be false, we may, if we have the slightest doubt of our accuracy, reëxamine the process of reasoning by which we were led to the false conclusion until there is no doubt of the accuracy of that process. If the process is found to be accurate, nothing remains to account for our false conclusion but the falsity of our original assumption.

The principle of logic illustrated by the above discussion may be stated as follows:

*If the conclusion of a correct process of reasoning be false, then the premises from which it necessarily follows must also be false.*[13]

**Summary of Principles Used in Indirect Proof.** Let us now summarize the principles of logic discussed above which are fundamental in all reasoning by the indirect method. These principles are as follows:

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[12] From Jevons, *Principles of Science*, pp. 5-6.

[13] Coffey, *The Science of Logic*, Vol. I, p. 339.

PRINCIPLE I. The Law of Excluded Middle. *A thing must either be or not be.*

PRINCIPLE II. *If one of two contradictory propositions is proved to be true, it immediately follows that the other must be false; similarly, if one of them is proved to be false, it immediately follows that the other must be true.*

PRINCIPLE III. *If the conclusion of a correct process of reasoning be false, then the premises from which it necessarily follows must also be false.*

**Definitions of Indirect Proof.** In view of the entire discussion up to this point, let us now redefine indirect proof with special reference to its use in geometry:

When the truth of a proposition is established by showing that to assume its contradictory as true leads us to a conclusion which is known to be false, the proposition whose truth is thus demonstrated is said to be proved *indirectly*, or by *reductio ad absurdum*. The conclusion resulting from assuming the contradictory proposition true is considered false or absurd if it is inconsistent with something previously accepted as true, that is, if it contradicts the given data, some axiom or postulate, or some previously proved theorem.

The following descriptions of indirect proof which are quoted from the works of some of the classic writers on logic and mathematics are worthy of careful study:

Aristotle (about 340 B.C.) refers to indirect proof by various terms such as *reductio ad absurdum*, *proof per impossibile*, or *proof leading to the impossible*. He describes it as follows: "Proof leading to the impossible differs from direct proof in that it assumes what it desires to destroy (namely the hypothesis of the falsity of the conclusion) and then reduces it to something admittedly false, whereas, direct proof starts from premises admittedly true." —From Heath, *The Thirteen Books of Euclid's Elements*, Second Edition, Vol. I, p. 136.

Proclus (about 450 A.D.), who wrote a commentary on Euclid, Book I, has the following description of the indirect method: "Proofs by *reductio ad absurdum* in every case reach a conclusion manifestly impossible, a conclusion the contradictory of which is admitted. In some cases the conclusions are found to conflict with the common notions (axioms), or the postulates, or the hypotheses (from which we started); in others they contradict propositions previously established. . . . Every *reductio ad absurdum* assumes what conflicts with the desired result, then, using that as a basis, proceeds until it arrives at an admitted absurdity, and, by thus destroying the hypothesis, establishes the result originally desired."—From Heath, *The Thirteen Books of Euclid's Elements*, Second Edition, Vol. I, p. 136.

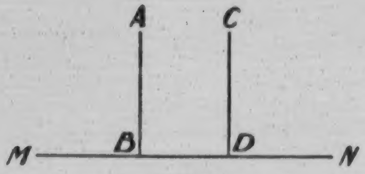
De Morgan (1847) defines indirect proof as follows: "When a proposition is established by proving the truth of the matters it contains, the demonstration is called *direct;* when by proving the falsehood of every contradictory proposition, it is called *indirect.*"—From De Morgan, *Formal Logic,* p. 24.

De Morgan also defines indirect proof in another one of his books as follows: "There are many propositions in which the only possible result is one of two things which cannot both be true at the same time, and it is more easy to show that one is *not* the truth, than that the other *is*. This is called indirect reasoning."—From De Morgan, *The Study and Difficulties of Mathematics,* p. 226.

Jevons (1877) gives this description: "The method of Indirect Deduction may be described as *that which points out what a thing is, by showing that it cannot be anything else.* In logic we can always define with certainty the utmost number of alternatives which are conceivable. The Law of Excluded Middle enables us always to assert that any quality or circumstance whatsoever is either present or absent. The Law of Contradiction is a further condition of all thought; it enables, and in fact obliges, us to reject from further consideration all terms which imply the presence and absence of the same quality. Now, whenever we bring both these Laws of Thought into explicit action by the method of substitution, we employ the Indirect Method of Inference. It will be found that we can treat not only those arguments already exhibited according to the direct method, but *we can include an infinite multitude of other arguments which are incapable of solution by any other means.*"—From Jevons, *The Principles of Science,* pp. 81-82.

**Applications of the Three Principles to Geometric Propositions.** Let us now apply the three principles of logic given on page 109 to one of the geometric propositions which is usually proved by the indirect method. In the following illustration, the comments in small type are not to be considered as a regular part of the demonstration.

PROPOSITION: *Two lines perpendicular to the same line are parallel.*



*Given:* Lines $AB$ and $CD$ each perpendicular to line $MN$.
*To Prove that:* Lines $AB$ and $CD$ are parallel.

*Proof:*

1. One of these propositions must be true: (1) lines *AB* and *CD* are parallel, or (2) lines *AB* and *CD* are not parallel.

   This follows from Principle I, which is the Law of Excluded Middle (page 109). Proposition (1) is the contradictory of proposition (2), hence one of these propositions is true and the other false.

2. Suppose the lines are not parallel.

   This assumes the contradictory proposition to be true.

3. Then *AB* and *CD* will meet at some point, such as *O*.
   Definition of parallel lines.
4. Then we would have two lines from a given point both perpendicular to the same line.
5. But the conclusion of step 4 is absurd, since it has previously been proved that from a given point only one perpendicular can be drawn to a given line.
6. Therefore, the supposition in step 2 is false.

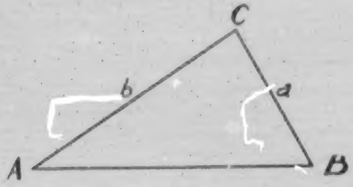
   This applies Principle III that if a correct process of reasoning leads to a false conclusion, then the premises from which it follows must be false (page 109).

7. Therefore, lines *AB* and *CD* must be parallel.

   The original proposition must be true, since it has been shown in step 6 that the contradictory proposition is false. This applies Principle II that if one of two contradictory propositions is proved to be false, the other must be true (page 109).

Let us now apply these principles to another typical proposition which is usually proved by the indirect method.

PROPOSITION: *If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.*



*Given:* Angle $B$ greater than angle $A$.
*To Prove that:* Side $b$ is greater than side $a$.

*Proof:*

1. One of these propositions must be true: (1) side $b$ is greater than side $a$, or (2) side $b$ is *not* greater than side $a$.

   This applies Principle I. Proposition (2) is the *contradictory* of proposition (1).

2. Suppose that $b$ is not greater than $a$. Then $b$ is either equal to $a$ or less than $a$.

   This assumes the contradictory proposition to be true. In this case, however, the contradictory proposition (namely, that $b$ is not greater than $a$) represents two different possibilities, namely that (1) $b$ equals $a$ or that (2) $b$ is less than $a$. It is necessary, therefore, to show that each of these possibilities leads to something false, in order to show that the *contradictory* proposition is false.[14]

3. Suppose side $b$ equals side $a$.

   This assumes one of the possibilities of the *contradictory* proposition to be true.

4. Then angle $B$ equals angle $A$. By previous proposition.
5. But the conclusion in step 4 is absurd, since it is given that angle $B$ is greater than angle $A$.
6. Hence the supposition in step 3 is false. Therefore, side $b$ is not equal to side $a$.

   This applies Principle III.

7. Suppose side $b$ is less than side $a$.

   This assumes the second possibility of the contradictory proposition to be true.

8. Then angle $B$ is less than angle $A$. By previous proposition.
9. But the conclusion in step 8 is absurd, because it is given that angle $B$ is greater than angle $A$.
10. Hence the supposition of step 7 is false. Therefore, side $b$ is not less than side $a$.

    This applies Principle III.

11. And it was proved in step 6 that side $b$ is not equal to side $a$.

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[14] See Heath, *The Thirteen Books of Euclid's Elements*, Vol. I, Second Edition, p. 140.

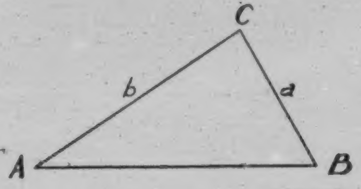
12. Therefore, side $b$ is greater than side $a$.

> The original proposition must be true since it has been shown in step 6 and step 10 that the two possibilities of the contradictory proposition are each false. This applies Principle II.

The rather full proof given above is for the purpose of illustrating further the principles of logic applied in a *reductio ad absurdum* proof; it is not intended as a model to be used in introducing this proof to high school pupils. For the latter purpose, it will be simpler, in general, to condense the proof somewhat, making the steps less formal, and arriving at the conclusion more quickly. If the pupil is required to dwell upon each separate step too long, he may lose the connection between steps and thus interrupt the drift of the argument.

**Euclid's Proof of the Above Proposition.** In introducing the proposition mentioned above to pupils for the first time, perhaps as clear and as simple a proof as can be found is the one originally given by Euclid. In fact, it is difficult to improve upon any of Euclid's original demonstrations so far as clearness and simplicity of language are concerned. A translation of the proof as it actually appears in Euclid's *Elements* is given below:[15]

PROPOSITION 19. *In any triangle the greater angle is subtended by the greater side.*



Let $ABC$ be a triangle having the angle $B$ greater than the angle $A$.

I say that the side $b$ is also greater than the side $a$.

For, if not, $b$ is either equal to $a$ or less.

Now, $b$ is not equal to $a$; for then the angle $B$ would also have been equal to the angle $A$ (I.5); but it is not;

therefore, $b$ is not equal to $a$.

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[15] The proof here given is taken from Heath, *The Thirteen Books of Euclid's Elements*, Vol. I, Second Edition, p. 284.

Neither is $b$ less than $a$, for then the angle $B$ would also have been less than the angle $A$ (I. 18); but it is not;

therefore $b$ is not less than $a$.

And it was proved that it is not equal either.

Therefore $b$ is greater than $a$.

Therefore, etc. Q. E. D.

It is seen that Euclid's proof follows the same general plan as the more detailed proof of this proposition given on pages 111 and 112, the third line of Euclid's proof corresponding to step 2 on page 112. In these particular steps of each proof, it should be noted that $b$ is assumed to be either equal to $a$ or less than $a$; this is merely another way of assuming that the contradictory proposition is true. In other words, in each proof the contradictory proposition admits of two possibilities, namely, that $b$ is either equal to $a$ or less than $a$, each of which is assumed to be true, and then proved to be false. In this way, the entire contradictory proposition is shown to be false.

**A Suggestion by De Morgan on Indirect Proof.** De Morgan has pointed out that in indirect proof it is sometimes difficult for the pupil to begin the actual argument by assuming formally that the contradictory proposition *is true* as was done in step 2 on page 112. As De Morgan expresses it, "It is rather embarrassing to the beginner to find that he is required to admit, for argument's sake, a proposition which the argument itself goes to destroy, but the difficulty would be materially lessened, if instead of assuming the contradictory proposition positively, it were hypothetically stated, and the consequences of it asserted with the verb *would be* instead of *is*."[16] It will be noted that the proof by Euclid given above, which was written over 2,000 years before De Morgan's time, has all these niceties of statement which De Morgan mentions. The third line of Euclid's proof (page 113) states the contradictory proposition hypothetically by the phrase *if not;* likewise, in both the fourth and the seventh lines of Euclid's proof the consequences of the assumption are asserted with the verb *would have been* instead of *is*. It may seem trivial to call attention to such detail, but it is refinements of this kind that do much to make indirect proof clear to pupils.

**Propositions Involving Three Relationships.** In each of the proofs discussed up to this point attention has been centered upon two major ideas, namely, that *a thing is true* or *it is not true*. These major ideas were represented in each case by the two contradictory propositions. Even in the cases of the propositions proved on pages 112 and 113 attention was first centered upon the fact that *b is greater than a* or *b is not greater than a*; we then subdivided the idea that *b is not greater than a* into the two possibilities that *b is equal to a* or *b is less than a*. In other words, the statement that *b is greater than a* considered the proposition *true*, while the statements that *b is equal to a* or *b is less than a* considered the proposition *not true*. Our object in bringing out these two major ideas was to make clear certain fundamental principles of logic which are essential in indirect proof.

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[16] De Morgan, *The Study and Difficulties of Mathematics*, p. 227.

In the propositions on pages 112 and 113, however, before the proofs were finally completed, we really had to consider three relationships in all, namely, that *b is greater than a, equal to a*, or *less than a;* but as was pointed out above, we did this by classifying these three relationships in two groups, one relationship constituting the *given* proposition, while the other two relationships constituted the *contradictory* proposition. Instead of treating these relationships in *two* groups, it would also have been possible, in the very first step of the proof, to have centered attention immediately upon *all three* relationships connecting *b* and *a*. Then our proof would have started by pointing out that there are three possibilities in all; namely, that *b is greater than a*, or *b is equal to a*, or *b is less than a*. We could then proceed to show that two of these possibilities lead to absurdities and hence the third possibility must be true. As an illustration of this treatment, all that we need to do is to delete the third line of Euclid's proof, as given on page 113, and replace it by the following: "Now *b* must be either greater than *a*, or equal to *a*, or less than *a*." The rest of the proof would be exactly as Euclid gives it.

This new method of approach, where all three possibilities are outlined in advance, avoids any direct reference to two contradictory propositions and proceeds on the basis of immediately laying all the cards on the table. Most of our modern textbooks in geometry handle this particular proposition in this same way. This procedure has much in favor of it, particularly for propositions where three possibilities are involved, and is probably the simplest approach when we have only a limited amount of time to devote to the indirect method. In this article we have purposely omitted discussing this particular method of approach until this time, since it is less suitable for illustrating the principles of logic involved in indirect proof. It should be pointed out, however, that both these methods of approach make use of the same principles of logic, and that both may be regarded as essentially alike; such differences as may exist between them are largely differences in organization of the initial steps of the proof.

**The Method of Elimination.** When a method of proof immediately centers one's attention upon all three possibilities, as was done above, we have a form of indirect proof that is sometimes called the *method of exhaustion* or *the method of elimination,* because the truth of one of the three possibilities is established by eliminating the other two possibilities. In such reasoning it is absolutely essential, if the final conclusion be valid, that *all* the possibilities be considered and that *all but one* of them be eliminated; the elimination of each of the possibilities (except one) is accomplished by showing that to assume the possibility true leads to a conclusion which is absurd, because it is contrary to certain known facts. The actual elimination of each possibility is accomplished, of course, by applying Principle III (page 109), namely, that if a conclusion of a correct process of reasoning be false, then the premises from which it necessarily follows must also be false.

The method of elimination as we encounter it in geometry is practically always limited to a total of three possibilities, but in life situations, where this method is often employed, there may be many more possibilities. The method of elimination leads to a dependable final conclusion, no matter how many possibilities are considered, provided we have all the possibilities in mind and also provided we can eliminate all but one of them.

Another distinction between the method of elimination as used in geometric propositions having three possibilities and this same method as applied to life situations is that in geometry, each of the possibilities eliminated is usually shown to be false because it leads to conclusions contrary to the given data; while in life, the various possibilities eliminated may be false because they lead to conclusions in violation of certain other observations, principles, or facts which have previously been shown to be true. Of course, the indirect proof given on page 110, where we proved that two lines perpendicular to the same line are parallel may be considered as a case of elimination where only two possibilities are involved; in this case, one of the possibilities was eliminated by showing th[illegible] it led to conclusions which violated a previous proposition, rather than the given data. It is evident that the method of elimination may be considered as another form of the proof known as *reductio ad absurdum*.

**The Law of Converse.** In geometry the method of elimination is most frequently employed to prove the converses of certain propositions. This leads us to consider what is known as the *Law of Converse*, which was first stated by Augustus De Morgan in his text on logic,[17] and which is proved by indirect reasoning. The Law of Converse may be stated as follows:

If the following three propositions have already been proved, namely, that

(*a*) If $X < Y$, then $A < B$.
(*b*) If $X = Y$, then $A = B$.
(*c*) If $X > Y$, then $A > B$.

then it follows logically that the converses of each of the above propositions must also be true, namely, that

(*a′*) If $A < B$, then $X < Y$.
(*b′*) If $A = B$, then $X = Y$.
(*c′*) If $A > B$, then $X > Y$.

In each of the above groups of propositions it should be noted that the three relationships connecting $X$ and $Y$ represent the only three possibilities that may exist and that one of these possibilities must be true. The same is true of the three relationships connecting $A$ and $B$.[18]

As an application of the above Law of Converse let $X$ and $Y$ represent two sides of a triangle and let $A$ and $B$ represent the angles opposite those sides; then, according to this law, the three converse propositions (*a′*), (*b′*), and (*c′*) are each immediately true if we have already proved that the three propositions (*a*), (*b*), and (*c*) are each true.

**Proof of the Law of Converse.** The proof of the Law of Converse is as follows: We know that propositions (*a*), (*b*), and (*c*) are true; hence proposition (*a′*), which states that if $A$ is less than $B$, then $X$ is less than $Y$, must be true. For, if $X$ is not less than $Y$, it must be equal to or greater than $Y$; but $X$ cannot be equal to $Y$, since by proposition ($b$) it would follow that $A$ equals $B$, which is impossible, since it is given that $A$ is less than $B$. Similarly, $X$ cannot be greater than $Y$ without violating proposition ($c$). Hence $X$ is less than $Y$, and proposition ($a'$) is true. The truth of propositions ($b'$) and ($c'$) may be established in a similar manner.

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[17] De Morgan, *Formal Logic*, p. 25.

[18] It is assumed that $A$ and $B$ are comparable with respect to magnitude; if $A$ were a real number and $B$ an imaginary number, the relations stated above would not hold. This assumption is also made for $X$ and $Y$.

**Other Forms of the Law of Converse.** The Law of Converse also applies if the original three propositions are stated as follows:

($a$) If $X < Y$, then $A > B$.
($b$) If $X = Y$, then $A = B$.
($c$) If $X > Y$, then $A < B$.

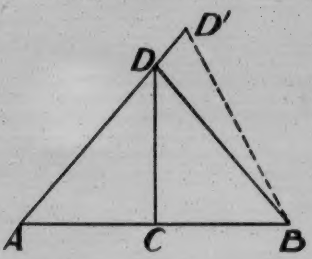
Hence it follows logically that the converse of each of these propositions is true. In the above form, the Law of Converse immediately applies to the propositions relating to chords of the same circle and their distances from the center of the circle; thus $X$ and $Y$ may refer to the lengths of two chords while $A$ and $B$ refer to the distances of these chords from the center of the circle.

A third form of the Law of Converse is illustrated by applying it to the Pythagorean proposition and its two related triangle theorems regarding the squares on the sides opposite an acute angle or an obtuse angle. From these three triangle theorems we may obtain the following statements, in which $S$ equals the sum of the squares of the other two sides:

($a$) If the angle is right, the square on the opposite side $= S$.
($b$) If the angle is obtuse, the square on the opposite side $> S$.
($c$) If the angle is acute, the square on the opposite side $< S$.

Hence the converse of each of these three statements is true.

Notice that the angle relationships stated above in ($a$), ($b$), and ($c$) represent the only three possibilities, namely, that an angle is acute, right, or obtuse.



The Law of Converse also applies in the following proposition:

*If DC is the perpendicular bisector of line AB, then all points on the bisector are equidistant from A and B, and all points not on the bisector are not equidistant from A and B.*

From this proposition we may easily establish the following three statements:

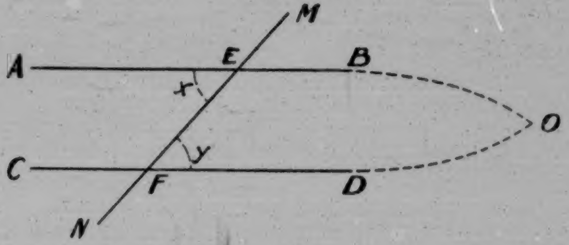
(*a*) If $D$ lies on the bisector, then $AD = DB$.
(*b*) If $D$ lies at the right of the bisector, then $AD > DB$.
(*c*) If $D$ lies at the left of the bisector, then $AD < DB$.

Hence the converse of each of these statements is also true.

Notice that there are only three possible positions for the point $D$, each of which is considered.

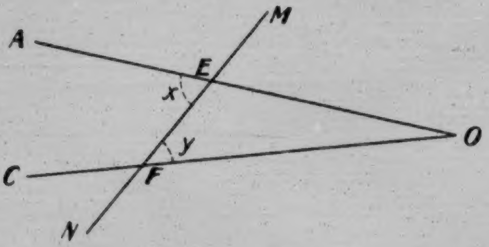
**Difficulties Caused by Distorted Figures.** Let us now discuss certain practical aspects of the teaching of the indirect method of reasoning. There is no question that this type of reasoning is difficult for pupils and also for teachers. Undoubtedly, some of the difficulty that pupils have with indirect proof is due to the fact that the figures for certain propositions are often distorted and misleading. The following proposition with its accompanying figure illustrates this point.

**Proposition:** *When two lines in the same plane are cut by a transversal, if the alternate interior angles are equal, the two lines are parallel.*



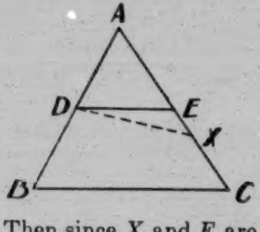
It is given that lines $AB$ and $CD$ are cut by a transversal $MN$ so that angle $x$ equals angle $y$. We are to prove that lines $AB$ and $CD$ are parallel. The proof of this proposition is commonly given by the indirect method, according to which we first assume lines $AB$ and $CD$ to meet on the right of $MN$ as at $O$, thus forming a triangle $EOF$. It would then follow that angle $x$ is greater than angle $y$, which contradicts the given data. Similarly, we assume $AB$ and $CD$ to meet on the left of $MN$. Since both these assumptions lead to absurdities, it follows that $AB$ and $CD$ are parallel.

The difficulty which the pupil experiences with this proposition is that the figure above suggests that the lines $AB$ and $CD$ can each bend so as to meet at $O$, but the pupil knows that this is impossible since $AB$ and $CD$ are straight lines which could not possibly bend as indicated. This difficulty could be avoided by omitting the dotted lines $BO$ and $DO$ in the above figure and by supplementing this figure by a second one like that given below. The given facts could



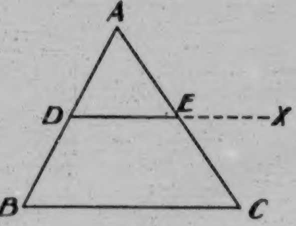
then be stated with reference to the first figure, while the assumption that the lines meet at $O$ could be referred to the second figure in which it is clearly seen that $EOF$ is a triangle. A third figure, which would show the lines meeting on the left of $MN$, might also be included.

The theorem that *the line joining the midpoints of two sides of a triangle is parallel to the third side* is another proposition that causes much confusion when an attempt is made to prove it by the indirect method. In the indirect proof, it is given that $DE$ joins the midpoints $D$ and $E$, and it is required to prove that $DE$ is parallel to $BC$. We first assume that $DE$ is *not* parallel, and that some other line through $D$, such as $DX$, *is* parallel to $BC$. Then $X$ must be the midpoint of $AC$ by a previous proposition. Then since $X$ and $E$ are both the same point, namely, the midpoint, we conclude that $DE$ must coincide with $DX$ and hence be parallel to $BC$, since only one straight line can be drawn through two given points. This last step is always unsatisfactory to the pupil.



It would be more satisfactory, if an indirect proof is to be used for the proposition in question, to omit that step in the above proof where we conclude that $DE$ and $DX$ must coincide and to say that if $DX$ is parallel to $BC$, then $X$ must be the midpoint of $AC$. But this is absurd since it is given that $E$ is the midpoint of $AC$. (A line cannot have two midpoints.) Hence the supposition that $DE$ is not parallel to $BC$ is false. Therefore, $DE$ is parallel to $BC$. Even this modified indirect proof is not satisfactory to many pupils, due to the fact that $DX$ is drawn in such a way that pupils can easily see that it is not parallel to $BC$, yet it is assumed to be parallel. They also feel that it is a waste of time to prove finally that $DX$ is not parallel to $BC$ when this fact is self-evident from the figure.

A still more satisfactory proof of the proposition under discussion is the one where we draw the figure without distortion as shown at the right. The proof runs as follows: $DE$ is given joining the midpoints $D$ and $E$. We are to prove that $DE$ is parallel to $BC$. Draw $DX$ through $D$ so that it will be parallel to $BC$. Then $DX$ bisects $AC$ (by previous proposition) and hence passes through the midpoint $E$. Hence $DE$ and $DX$ must coincide since both pass through the same two points. (Only one straight line can be drawn through two given points.) On careful examination it will be seen that it would be more correct to classify this demonstration as a direct proof rather than indirect.[19]



**Difficulties Due to Early Presentation of Indirect Proof.** One other cause for the difficulty that the pupil experiences with indirect proof is related to the fact that this type of proof is presented too early in the course in demonstrative geometry, before the pupil has scarcely begun to understand the nature of direct proof. When the first indirect proof comes up, it is usually presented abruptly without any preparation whatever for this type of reasoning. The result is that the pupil forms unfavorable impressions concerning such proof that are more or less permanent. In view of this situation, it seems to be pedagogical common sense to delay the introduction of indirect proof until the pupil has become quite familiar with direct proof.

I well realize, however, that the sequence of propositions given in Book I of most geometries requires the introduction of indirect proof fairly early. There seems to be no way to avoid it when we reach the proposition that if a transversal cuts two lines so as to make a pair of alternate angles equal, the lines are parallel. Indirect proof is also frequently used to prove the converse of this proposition. It seems to me that it would greatly simplify matters to assume these two parallel propositions as true without going through the intricacies of indirect proof. Certain other early propositions which are proved indirectly might also be assumed. A little later in the course, after the pupil has a fairly thorough understanding of direct proof, we could then introduce indirect proof, going back and proving the propositions whose truth had previously been assumed.

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[19] Due to the dissatisfaction caused by the indirect proofs discussed above, which relate to the proposition concerning the line joining the midpoints of two sides of a triangle, many modern texts in geometry are using a simple direct proof of this proposition which makes no mention whatever of the final coincidence of the two lines $DE$ and $DX$; this is probably the wisest plan of all.

In this connection, I should mention that there is considerable discussion to-day regarding the advisability of introducing a unit in demonstrative geometry in the third year of the junior high school. So far, comparatively few schools have had much experience in teaching such a unit. I am convinced that if a unit in demonstrative geometry is to be generally successful in the third year of the junior high school, its success will depend upon a marked simplification of the syllabus of propositions offered for that unit. I believe we can safely say that an important element in this simplification will be *to assume as true* those propositions which require indirect proof, reserving for the first year of the senior high school a thorough study of the indirect method.

**Difficulties Due to Insufficient Practice.** Still another reason why the pupil finds indirect proof difficult, which was mentioned earlier in this article, is that indirect proof is used so little in the entire course in geometry that the pupil does not get enough practice in applying it. In most all our textbooks in geometry, about 95 per cent of the theorems and exercises are proved by direct methods. Further, such exercise as the pupil does get in indirect proof is largely in Book I at a time when his short acquaintance with geometry finds him least prepared to understand it. The only solution to this difficulty seems to be for teachers to recognize the fact that a mastery of this kind of reasoning will come only after the pupil has solved a large number of exercises making use of it and has come to understand the fundamental nature of indirect proof as clearly as he understands that of direct proof. It is a fundamental principle of teaching that the mastery of a new idea or a new way of thinkin[illegible]ike the learning of a skill, comes only through sufficient repetit[illegible] and practice; the mastery of indirect reasoning is no exceptio[illegible] this principle.

**Indirect Proof as [illegible]ype of Analysis.** In connection with the difficulty just dis[illegible]d, it is of interest to note that teachers have been somewhat [illegible]onsistent in their great emphasis upon direct proof and their comparative neglect of indirect proof. Most teachers are devoting a considerable amount of time in the geometry classes to-day in making clear that *analysis* is a fundamental method of discovering direct proofs. It is surprising, however, that these same teachers are giving practically no time at all to an explanation of the nature of indirect proof, though "it too is merely *a case of analysis* in which the subsequent synthesis, that is usually required as a complement, may be dispensed with."[20] Looking at this in another way, "analysis is nothing else but a method of reduction."[21] If analysis is so important, as all agree that it is, why limit the practice of it almost exclusively to direct proof?

The fact that analysis is fundamental in each type of proof may be seen more readily by comparing the usual procedure for discovering a direct proof with that employed in obtaining an indirect proof. These procedures are as follows:

(1) To obtain the *direct proof* of a proposition by the method of analysis, we assume the *given proposition* to be *true* and then work *back* until we arrive *at the given data.* We then *reverse* the order of these steps to obtain the direct synthetic proof.

Let fact $C$ represent the conclusion of the given proposition which is assumed to be true. In working back to the given data, we reason as follows: Fact $C$ is true if fact $B$ is true; fact $B$ is true if fact $A$ is true; but fact $A$ is true because it represents the given data. Hence, by reversing the order of these steps we obtain the synthetic proof.[22]

(2) To obtain the *indirect proof* of a proposition, we assume the *contradictory of the given proposition* to be true and then work *forward* until we arrive *at something that conflicts with the given*

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[20] Allman, *Greek Geometry from Thales to Euclid*, p. 140. See also Heath, *The Thirteen Books of Euclid's Elements*, Vol. I, Second Edition, p. 140.

[21] "L'Analyse n'est donc autre chose qu'une méthode de réduction."—Duhamel, *Des Méthodes dans les Sciences de Raisonnement*, première partie, 3rd Edition, p. 41.

The method of analysis might also be called the *method of reduction* or the *method of successive substitutions*.

[22] For a full discussion of analysis and synthesis, see Heath, *The Thirteen Books of Euclid's Elements*, Second Edition, Vol. I, pp. 137-142.

*data* (or some other previously established or accepted fact). It then follows *without reversing* the steps that the contradictory proposition is false and hence that the given proposition is true.

Let fact *R* represent the conclusion of the contradictory proposition which is assumed to be true. In working forward, to a step which conflicts with the given data (or some other known fact), we reason as follows: If fact *R* is true, then fact *S* is true; if fact *S* is true, then fact *T* is true; but fact *T* cannot be true since it conflicts with the given data (or some other known fact). Hence, the contradictory of the given proposition is false; and the given proposition is true.

It is thus seen that *analysis* is as characteristic of indirect proof as it is of direct proof. Hence, to be consistent, we should give far more time than we do now in seeking opportunities to apply analysis in indirect proof. The result of such a practice would do much to assure a mastery of the method of *reductio ad absurdum* and thus remedy one of the prominent difficulties mentioned above.

**A Thorough Treatment of Indirect Proof.** To instruct the pupil thoroughly in indirect proof will require about five to ten times as much time and effort as is ordinarily given to this type of proof in high school classes to-day. All textbooks in geometry and most teachers of this subject now give very little attention to this topic. For a *thorough treatment* of the subject, it is really necessary to give the pupils in very elementary form an acquaintance with certain simple notions of logic. A professor of logic in one of our large universities recently deplored the fact that logic is no longer a required subject in our college courses, whereas some thirty years ago, logic was practically always prescribed for college students. This logician seemed to feel that the only solution would be to introduce the teaching of elementary logic in our high schools. While it may be many years before such a hope is realized, it does seem essential that certain fundamental notions of logic should be discussed in the geometry class preparatory to the study of indirect proof. The notions I have in mind are simple enough and, provided they are given sufficient repetition, the pupils should easily get them.

These simple notions are really the three principles of logic which were stated on page 109 of this chapter. It is not necessary that these principles should be stated formally by the pupils, but it is essential that the ideas involved in them should be appreciated.

The first important idea to be made clear to the pupils is that every proposition is true or it is not true. By careful questioning, pupils will grant that Mary either has a pair of skates or she does not have a pair of skates; that John either weighs 100 lbs. or he does not weigh 100 lbs.; that angle *A* either equals angle *B* or it does not equal angle *B*. Thus, in a simple way, by repeated illustrations, we can make clear Principle I, which is stated on page 109.

The pupil is now ready to see that the statements just made are of such a nature that one *contradicts* the other, because one says that a thing *is so* while the other says that it *is not so*. We then tell the pupil that such statements are called *contradictory propositions*. Numerous examples of contradictory propositions should be given, not only examples of contradictory geometric propositions, but also examples of contradictory life propositions.

We are now ready to point out that if we have two contradictory propositions, one must be true and the other must be false. If we prove that one of them is true, it follows that the other must be false. Likewise, if we prove that one of them is false, it follows that the other must be true. By a certain informal discussion of this idea, we make clear Principle II, as given on page 109.

We are then ready to present some of the simpler geometric propositions which are proved by the indirect method, such as the one given on pages 110 and 111. At first the proposition should be presented much as it is usually done, without any reference to the logical principles that are applied. Then the proposition can be repeated, calling attention in the various steps of the proof to the fact that we have used the ideas developed above. After several such proofs, we can then probably best present Principle III, which we have been using in our proofs, but which we do not point out definitely until now.

Undoubtedly the best way to get pupils to appreciate the nature of indirect proof is not first to state all the logical principles involved and then to apply them in proofs, but to alternate, first presenting a simple proof, next pointing out the principles applied, and then, in turn, illustrating these principles by further proofs.

In this thorough treatment of the indirect method propositions involving three possibilities will be introduced in much the same way as they were presented on pages 112 and 113, starting with the simpler form of the proof given on page 113 and then pointing out the principles applied as on page 112. After this is mastered, then the method of elimination may be presented and finally coördinated with the treatment given on page 112, the relation between contradictory propositions and the method of elimination being discussed as on pages 115 and 116. In all this work, applications of the indirect method, including the method of elimination, will be made not only to geometric propositions, but also to life situations. If teachers are willing to spend enough time to give such a thorough and detailed presentation of indirect proof and its principles, I am sure they will be repaid for their efforts.

**A Brief Treatment of Indirect Proof.** If teachers cannot spare the time necessary to give a very full treatment of indirect proof as outlined above, I will suggest the following briefer treatment of this topic. Probably the simplest approach will be to start with a proposition like the one given on page 110, having the pupils go through the various steps of the proof without reference to the principles of logic involved. It will be helpful in this presentation to point out for a proposition like that on page 110 that only two possibilities are involved, namely, that the lines are parallel or that the lines are not parallel. Show the pupil that if one of these possibilities leads to an absurdity, then the other must be true. The next step will be to proceed to the method of elimination, giving a proposition involving three possibilities in all, these three possibilities being pointed out at the very beginning of the proof as was suggested on page 115. This means that in discussing the method of elimination no reference will be made whatever to contradictory propositions; in fact, contradictory propositions will not be mentioned at all in this briefer treatment.

In teaching the method of elimination along the lines just indicated, where all three possibilities are brought to the pupil's attention at the beginning of the proof, it will be found that pupils often ask how to tell which one of the three possibilities is to be left until the last. In other words, they want to know how to tell which two of the three possibilities are to be eliminated. The answer to this is that we keep until the last the possibility which we wish to prove true; in other words, we keep until the last the possibility which is represented in the statement of the given proposition, eliminating the other two possibilities. The question as to which possibility to keep to the last is not likely to arise in the more thorough treatment of indirect proof, outlined above, where the fundamental principles are studied and where the whole treatment is related to the idea of contradictory propositions. In the briefer treatment, pupils will be interested to see the method of elimination also applied to life situations.

All our high school textbooks in geometry give what I would call the briefer treatment of indirect proof. In most of these books, however, the pupil is given simple directions for attacking exercises by the indirect method. Probably the simplest directions of this type are *to assume, for the sake of argument, that the conclusion of the given proposition is not true and to show that this assumption leads to an absurdity.* It should be pointed out that these directions amount to the same thing as assuming that the contradictory proposition is true and showing that this assumption leads to an absurdity (which amounts to proving the contradictory proposition to be false). In our briefer treatment of the indirect method, however, we cannot speak of contradictory propositions. Hence the directions given above in italics will be the simplest that we can use in the absence of a knowledge of contradictory propositions. It will be found that these directions are particularly serviceable where the exercises to be proved involve only two possibilities. They are likewise serviceable in cases involving three possibilities, but for such cases, if we are giving a brief treatment of the indirect method, it will probably be simplest first to pick out the three possibilities and then to proceed to eliminate.

In making the above suggestions for a brief treatment of the indirect method, I do so merely as a compromise in case the teacher can afford only a limited amount of time to devote to the study of this topic. On the other hand, I strongly recommend the more thorough treatment of indirect proof, which includes a study of certain principles of logic, as the most desirable course to follow.

**Importance of the Study of Indirect Proof.** My aim in urging this more elaborate instruction in indirect reasoning is not for the sole purpose of enabling the pupil to understand the few indirect proofs which he usually encounters in his work in geometry. I have also in mind to enable him better to appreciate indirect proof as it is applied in life situations. After the pupil has finished his study of geometry, he will still have many opportunities to use indirect proof in life. If Mr. Jevons' statement is correct that "nearly half of our logical conclusions rest upon its employment,"[23] then the time spent upon indirect proof in the geometry class will be an investment of permanent value.

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[23] Jevons, *The Principles of Science*, p. 82. It is important not to misinterpret Mr. Jevons' statement. He says that nearly half of our logical conclusions depend upon indirect reasoning, but this should not be understood to mean half of all our thinking; we do much thinking that does not result in logical conclusions.

I should add also that I am convinced that instruction in indirect proof of the type I have outlined will carry over into life situations; in other words, I believe in the transfer of training in the case of a study like demonstrative geometry. Apparently the educational psychologists have given up their severe attacks upon the theory of formal discipline, which have been so frequent during the past twenty years, and are now admitting that certain school subjects *do have* disciplinary value and that transfer *does* take place.[24] Hence there is more hope than ever that skill in indirect reasoning gained in the geometry class will also function in the affairs of everyday life. To make certain that it will function, however, the instruction must be of the right kind.

**Illustrations of Indirect Reasoning in Life.** In order to emphasize the fact that indirect reasoning does occur in life, a few illustrations of its use in everyday situations will now be presented. One such illustration, describing an attempt to tell whether a steam valve was defective or not, was given at the beginning of this article. The additional illustrations given below represent situations that have actually occurred.

First, let us consider the legal device by which an accused person shows that he is innocent by proving an *alibi*, this being an excellent illustration of indirect reasoning. For example, suppose a man is accused of stealing a watch from Room 10 of the Adams Hotel between 7 and 8 P.M. on November 15. He proves his innocence by showing that he was dining at a friend's home five miles distant from the Adams Hotel during the entire hour when the theft occurred.

The indirect nature of this proof is easily seen by putting it in steps, similar to those used above in geometric proofs:

PROPOSITION: *John Doe (the accused) is innocent of the crime of stealing a watch between 7 and 8 P.M. on November 15 from Room 10 of the Adams Hotel.*

*Given:* Details of the disappearance of the watch from Room 10 of the Adams Hotel between 7 and 8 P.M. on November 15.

*To Prove that:* The accused is innocent of the crime.

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[24] See Orata, *The Theory of Identical Elements*, Chaps. 9 and 10. See also Gates, *Psychology for Students*, Revised Edition, Chaps. 11 and 13.

*Proof:*

1. One of two things is true: (1) the accused is innocent, or (2) the accused is guilty.
2. Suppose the accused is guilty (this assumes the contradictory proposition to be true). Then, he must have entered Room 10 of the Adams Hotel between 7 and 8 P.M. on November 15. But this is absurd since the accused was dining at a friend's home, five miles distant from the Adams Hotel during the entire period between 7 and 8 P.M. on November 15.
3. Hence the supposition in step (2) is false. Therefore, the accused is innocent.

A second illustration is that of Henry Jones, a child about ten years old, who had spent many hours in making a pretty calendar which he presented to his mother as a Christmas present. His mother appreciated the gift and placed it upon her dresser. A few days after Christmas, Henry's Uncle John came to visit the family and the mother showed him all her Christmas presents except the calendar. The child noticed the omission but said nothing. A day later, the child went to his mother's room, took the calendar, and tore it into bits. He then picked up the pieces, took them to a room where his mother was sitting, and threw them into her lap, saying, under great emotional strain, "There's your calendar, you didn't like it anyway!" The mother was greatly surprised and asked Henry why he did that. He replied, "Well, you didn't like it, because you didn't show it to Uncle John yesterday when you were showing him your presents!" This is a good example of indirect reasoning, which readily reduces to two contradictory propositions, namely, that (1) the mother likes the calendar, or (2) the mother does not like the calendar. The child reasoned that if the mother had liked the calendar she would have shown it to Uncle John along with her other presents, but she didn't do this, hence she did not like the calendar.

A third illustration relates to an experience of Mr. Brown who owned three cars, a Cadillac, a Buick, and a Ford, which he kept in a garage at the rear of his residence. As he was leaving home one morning to go to his office, he told his son Tom that he wanted him to drive to Meadeville, some 40 miles away, that afternoon to do an errand for him, to which Tom agreed. During the afternoon, Mr. Brown telephoned home and asked the maid what car Tom had taken when he left for Meadeville. The maid replied that she did not know, since she did not see Tom drive out. She said she knew that Mary had taken the Ford about a half hour previously to go to a tennis match. Mr. Brown asked the maid to step out to the garage to see what car was there; the maid reported that the Cadillac was the only car left in the garage. Hence, Mr. Brown decided that Tom was driving the Buick.

A fourth illustration is that of two men taking a walk in the woods, within a hundred miles of New York City, on a day in early June. One heard an insect chirping and said, "I wonder if that is a cricket." The other replied, "They say that frost comes six weeks after one hears the first crickets of the season." To which the first instantly replied, "Then that isn't a cricket." This argument is an illustration of reasoning by elimination, which is, of course, indirect proof. In this case, the possibility that the insect could be a cricket was eliminated since it was assumed as true that frost comes six weeks after the first crickets of the season. This argument, however, did not establish what kind of an insect it was, since all the other possibilities were not available for consideration. In life, however, it is often quite as important to prove that a thing *is not* true as it is to prove that something else *is* true.

As a final illustration, let us consider an experience of one of my friends who had been suffering from attacks of pain which seemed to originate in the stomach. He consulted a specialist in diseases of the stomach who proceeded to determine the cause of the pain. The physician reasoned that, according to his experience, pain in the stomach of the particular kind from which the patient was suffering might be due to one of four causes: (1) gall stones, (2) an ulcer of the stomach or the intestines, (3) adhesions about the stomach or the intestines, or (4) a chronically inflamed appendix. He made the usual physical examination, but that did not give information of a sufficiently definite kind to point to any of these factors as the probable cause of the trouble. The physician then ordered a series of x-ray plates which showed that no ulcer was present and that no adhesions were present; hence, items (2) and (3) were eliminated as possible causes of the pain. Another special test combined with a second x-ray examination indicated that there were no gall stones; hence, item (1) was eliminated. This left the chronically inflamed appendix as the only remaining possibility, and he told the patient that he believed that that was the cause of his trouble. This surprised the patient, because he said he had never experienced any soreness or discomfort in the region of the appendix, and even during the physical examination which had just been made there was no tenderness whatever about the appendix. In spite of the absence of local symptoms, the physician said that it was his firm belief that a chronically inflamed appendix was the cause of the pain in the stomach. He pointed out that during the attacks of pain the inflamed appendix was making repeated efforts to rid itself of the accumulated pus, thus inducing spasms which in turn traveled upward along the intestinal tract, causing sympathetic spasms in the stomach, and that it was the spasms of the stomach that produced the pain that was felt; these spasms caused no pain around the appendix because the appendix is so small, but in a large organ like the stomach such spasms did produce pain which was caused by the violent muscular contractions which were set up in that organ. In other words, it was a case of the tail wagging the dog. The physician advised my friend to have an operation for the removal of his appendix. The advice was followed and the surgeon discovered a chronically inflamed appendix, which had probably been in that condition for many years. A few weeks after his recovery from the operation, the patient's pain had disappeared.

**Reasons for Teaching Demonstrative Geometry.** In closing this discussion, I wish to state that I firmly believe that the reason we teach demonstrative geometry in our high schools to-day is to give pupils certain ideas about the nature of proof. The great majority of teachers of geometry hold this same point of view. Some teachers may at first think that our purpose in teaching geometry is to acquaint pupils with a certain body of geometric facts or theorems, or with the applications of these theorems in everyday life, but on second reflection they will probably agree that our great purpose in teaching geometry is to show pupils how facts are proved.

I will go still further in clarifying our aims by saying that on the part of the more progressive teachers to-day, the purpose in teaching geometry is not only to acquaint pupils with the methods of proving geometric facts, but also to familiarize them with that rigorous kind of thinking which Professor Keyser has so aptly called "the If-Then kind, a type of thinking which is distinguished from all others by its characteristic form: *If* this is so, *then* that is so."[25] Our most famous model of this kind of thinking is Euclid's *Elements*, but, as Professor Keyser has so clearly pointed out, Euclid's great contribution was not to geometry but to *a method of thinking* which is applicable not only in mathematics, but also "in every other field of thought—in the physical sciences, in the moral or social sciences, in all matters and situations where it is important for men and women to have logically organized bodies of doctrine to guide them and save them from floundering in the conduct of life."[26]

The fact that the teaching of proof and deductive thinking are our main objectives in the teaching of demonstrative geometry in the tenth year of the high school is much more true to-day than it was twenty-five years ago, before the junior high school movement started. With the development of the junior high school came the introduction of work in intuitive geometry in the seventh and eighth years where the main concern is to familiarize pupils *experimentally* with certain important geometric facts and their applications in everyday life. Hence, in the senior high school, we are primarily interested in teaching pupils something new rather than in reviewing those things that they have already learned in intuitive geometry. Our great aim in the tenth year is to teach the nature of deductive proof and to furnish pupils with a model for all their life thinking. Everyday reasoning will be rigorous and conclusive to the extent to which it approaches that ideal pattern for thought that Euclid has given us.

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[25] Keyser, *Thinking About Thinking*, p. 18. This interesting little book of about 90 pages should be read by every teacher of geometry; it will do much to stimulate the imagination of teachers with reference to the possibilities of relating the study of geometry to the thinking of life.

[26] Keyser, *Thinking About Thinking*, p. 35.

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# THE ANALYTIC METHOD IN THE TEACHING OF GEOMETRY

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## I. The Analytic Method and the Logic of Geometry

**Why We Teach Demonstrative Geometry.** Modern educators justify the presence in the school curriculum of geometry or any other school subject on two grounds: first, that it gives the student an outlook upon a great field of human thought; second, that, if properly taught, it gives him valuable habits of thinking and ideals of method which have transfer value and which help him in orderly and systematic thinking. Geometry seems, of all secondary school subjects, best adapted to initiate a student into the meaning of mathematics as the science of necessary conclusions. A training in logical processes and a knowledge of what is meant by a complete proof of a proposition, based on given data, are as necessary for the average citizen as a study of economics or civics. How can he draw valid conclusions in these fields unless he has had some training in logical processes? Geometry furnishes the best available material for this training, varying from the simplest to the most complex; starting with a few assumptions or axioms, and building a logical system which results in a body of established truths which can be used to establish further truths. In geometry more than in any other school subject, the learner is led to a belief in reason, and is made to feel the value of a demonstration. The appeal is to the authority of logically established propositions, running in a series back to the simple and accepted axioms, and not to arbitrary authority.

This attitude toward the value of mathematics is taken by both modern educators and the world's greatest mathematicians. Smith and Reeve in their book say [1] that the real purpose of demonstrative geometry is suggested by the word "demonstrative" rather than by the word "geometry." In geometry we come in contact with a body of truths bound together and unified by logical processes into a perfect whole. The beauty, purity, and perfection of the subject enable us to realize what it is to function in an ordered cosmos, where the reign of law is absolute and where the series of situations and problems presented vary from the simple and easy to the most difficult. When a student is engaged in the analysis necessary to discover the demonstration of a theorem in geometry, or in reasoning out a construction problem, he has entered this ordered cosmos and is tasting the delights of pure reason. He is becoming acquainted with the inner nature of that vast body of human thought, built up through the ages.

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[1] Smith, David Eugene and Reeve, William David. *The Teaching of Junior High School Mathematics*, p. 229. Ginn and Co., 1926.

**The Analytic Method of Attack.** But to be truly initiated into the spirit and meaning of logical processes, he must acquire the mastery that comes only from analytic thinking. The mere committing to memory of theorems and their demonstrations is not mastering geometry in this sense. The student has not reached this mastery until he has learned by analytic processes to discover proofs for himself and to assemble them in elegant deductive arguments. Only when he has received sufficient training to attack confidently a new proposition or problem, starting at the goal or conclusion, working backward step by step to his previously established body of truth, can he be said to have mastered the spirit of geometric reasoning. He must be able to say:

> "I can prove $X$ if I can prove $Y$,
> And I can prove $Y$ if I can prove $A$,
> But I can prove $A$ if $B$ is true.
> I have already proved $B$; hence I can prove $X$."

Then he must be able to reverse the process in his synthetic demonstration, starting with $B$ and proceeding through $A$ and $Y$ to $X$, which was to be proved.

**Analysis a Method of Discovery.** Analysis is the method of discovery, and *the only method of organizing the subject matter of geometry which gives sufficient command of the logical processes to justify its study.* In many of our schools, even to-day, theorems are committed to memory and recited by the pupils. A student does not see why each statement in the proof is made, although he may see that it is true, and may follow the meaning and admit the truth of each statement, without seeing how the author knew what statement to make next. He seems to be led step by step into a trap, and the trap sprung at the end with the Q.E.D.; but he himself is not able to lay such a logical train, because he has not been taught to analyze, working his way from the unknown, or truth to be demonstrated by logical steps, to his previously established body of truth. Such training takes time, and progresses slowly at first. But the mastery thus acquired enables a class to make up for lost time in the second half of the term's work, and the consequent joy in the work that comes with the sense of mastery, the consciousness of the pure and austere beauty of the subject that comes with this intimate mastery of the logical processes, more than compensate for the time and effort required.

**Value of Geometric Training.** Geometric training is valuable if it gives:

1. Clear geometric concepts through drawing, measurement, experiment, in the early stages.
2. A clear concept of and practice in logical proof, cast in the synthetic form.
3. Training in the analytic method of attack for the discovery of such synthetic demonstrations.
4. An ability to resolve practical problems into the geometric elements involved, and to solve them *by analysis*.

The analytic method is thus the heart of geometric work. A student thus trained should be able to originate proofs of his own, different from those of the text, and should be encouraged to originate additional proofs. His discovery of proofs for originals must not be a blind groping for a proof, following remembered models and analogies, but must be a consistent, confident, systematic, analytic attack; otherwise he has neither mastered geometry nor acquired the most valuable fruit of logical training, an ability to analyze.

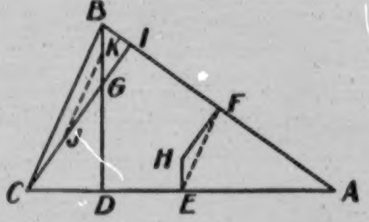
## II. EXAMPLES OF THE ANALYTIC METHOD

**Illustrative Examples.** To make the above generalizations clear, it may be well to take specific examples of the three principal types of logical exercise in geometry—a theorem to be demonstrated, a construction to be performed, and a problem (of computation) to be solved—and trace the steps of analysis that a student reasonably well trained in this method might be expected to take.

**Original Theorem.** As an example of an *original* theorem, let us take the following:

*If the altitude $BD$ of $\triangle ABC$ is intersected by the altitude $CI$ in $G$, and $EH$ and $HF$ are perpendicular bisectors of $AC$ and $AB$, respectively, prove that $BG = 2HE$ and $CG = 2HF$.*

The first step called for is to translate the words of the theorem into a figure similar to this figure:



**Analysis.**

1. I can prove that $CG = 2HF$ and $BG = 2HE$, if I can prove that $\frac{1}{2}CG = HF$ and $\frac{1}{2}BG = HE$, or if I can double $HF$ and $HE$ and then prove that the doubles are equal to $CG$ and $BG$, respectively. Let us try the first method. This suggests bisecting $CG$ and $BG$ at $J$ and $K$.
2. I can prove that $JG = HF$ and $KG = HE$, if I can prove that they are corresponding parts of congruent triangles. This suggests drawing $JK$ and $EF$, forming $\triangle JGK$ and $HEF$.
3. I can prove these triangles congruent if I can show that three parts of one are equal respectively to three parts of the other in certain orders.
4. Therefore I survey the figure and find:
   (a) That $JK$ and $EF$ are both equal to $\frac{1}{2}BC$ and parallel to $BC$, as they join the midpoints of the side of the $\triangle CGB$ and $CAB$, respectively.
   (b) That $\angle G = \angle H$, as they have their sides parallel, and extending in opposite directions from their vertices.
   (c) That $\angle K = \angle E$ for the same reason.
5. Therefore $\triangle JGK \cong \triangle HEF$ because *s.a.a.* = *s.a.a.*

The analysis is now complete, but the student should cast the proof into the elegant and convincing form of the synthetic presentation, giving statements and reasons in strictly logical fashion. The analysis outlined above may be carried out coöperatively by the class and the teacher when difficult originals are developed, but the work must be done heuristically. The teacher should ask such questions as the following.

"How do we usually prove that one line is the double of another?" The substance of step 1 in the analysis above should come from the class in response. When the lines $JK$ and $EF$ have been drawn, the question should be:

"How do we usually prove that a line equals another line?" and in response the class should announce step 2 above. True heuristic teaching will develop in the class the habit of asking *themselves* the appropriate questions of the analysis, and this is greatly helped by developing *outlines of methods of attack as the term progresses.*

**Synthesis.** The synthetic proof reverses the order of the steps given above in the analysis. Thus we proceed in the synthetic proof as follows:

**Proof**

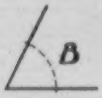
| **Statements** | **Reasons** |
|---|---|
| Bisect $CG$ and $GB$ at $J$ and $K$, respectively. Draw $JK$ and $EF$. Then in the triangles $JGK$ and EFH, | |
| 1. $JK = \frac{1}{2}CB$ and is $\parallel$ to $CB$ | 1. A line joining the midpoints of two sides of a triangle is $\parallel$ to the third side and equal to half the third side. |
| 2. $EF = \frac{1}{2}CB$ and is $\parallel$ to $CB$ | 2. Same reason. |
| 3. $\therefore JK = EF$ and $JK$ is $\parallel$ to $EF$ | 3. Being equal to half and $\parallel$ to the same line. |
| 4. Also $BD$ is $\parallel$ to $HE$ and $CI$ is $\parallel$ to $HF$ | 4. Being $\perp$ to the same straight lines. |
| 5. $\therefore \angle G = \angle H$ | 5. Because their sides are $\parallel$ and extend in opposite directions from their vertices. |
| 6. Also $\angle K = \angle E$ | 6. Same reason. |
| 7. $\therefore \triangle JGK \cong \triangle HEF$ | 7. s.a.a. = s.a.a. |
| 8. $\therefore JG = HF$ and $GK = HE$ | 8. Corresponding parts of $\cong$ ▲ are $=$. |
| 9. And hence $GC = 2HF$ and $GB = 2HE$ | 9. Doubles of equals are equal. |

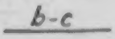
Until the student can go through the analytic process outlined above mentally, when presented with a new theorem to be proved, and then proceed to write the synthetic proof, he has not acquired a sufficient mastery of the logical processes involved in geometry to give him that acquaintanceship with logical reasoning at which we aim in teaching him the subject.

**Construction Problems.** Construction problems should never be solved by the student by an unguided trial-and-error method. He should first draw a diagram, representing approximately the finished product. Even a free-hand drawing of the finished job will enable him to mark on his figure of analysis the given parts, and to discover from the *diagram of the finished figure* the steps necessary to build up that figure from the given parts. Let us take a specific example:

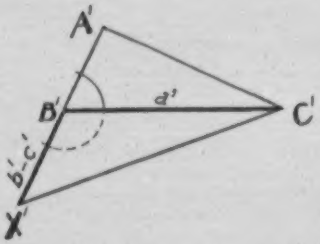
*Construct a triangle, having given an angle, an adjacent side, and the difference of the other two sides.*

Using the usual method of designating the parts of a triangle, we have given,



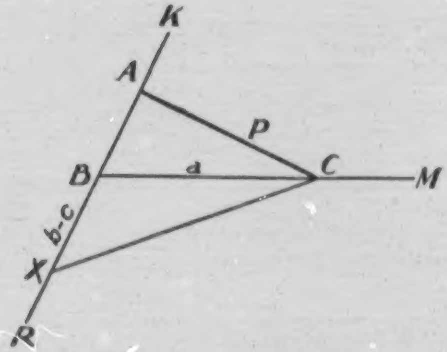


**Analysis.** We draw a triangle, $A'B'C'$, to *represent* the finished figure, thus:



Since the given parts must all be represented in our figure of analysis, we extend $A'B'$ to $X'$, making $A'X' = A'C'$. Then, $B'X'$ *represents* the difference of $b$ and $c$ and is marked $b' - c'$ on the diagram of analysis. We now reason analytically thus:

1. I can reproduce the figure, which must be like this model (using the given parts), if I can find a triangle in the model containing three known parts. I therefore examine the figure of analysis to find this basis of construction.
2. I notice that $\triangle X'B'C'$ contains two known sides, $b' - c'$ and $a'$, representing $b - c$ and $a$, respectively. Also $\angle X'B'C'$ is the supplement of $\angle B'$, and is therefore known.
3. I can therefore reproduce $\triangle XBC$, using the *given parts*. The construction can be completed if I can reproduce the rest of the figure from this basis. Therefore I examine the model.
4. Prolonging $XB$ will give a line corresponding to $X'B'A'$. To locate the point $A$, I notice that $\triangle A'X'C'$ is isosceles. Therefore $\angle X'C'A' = \angle X'$. Hence, on the figure to be constructed, I must cut the line $XB$ produced by a line drawn at point $C$, making $\angle XCA = \angle X$. The construction then proceeds thus:



**Construction.** On any indefinite line $BM$, lay off $BC = a$. At $B$ construct $\angle KBC$ equal to the given $\angle B$, giving the indefinite line $RK$. On $KR$ from $E$, construct $BX$ equal to $b - c$. Draw $XC$. At $C$ construct $\angle XCP = \angle X$. Extend $CP$ to meet $RK$ at $A$. Then $\triangle ABC$ is the required triangle.

**Proof**

| **Statements** | **Reasons** |
|---|---|
| 1. $BC = a$ | 1. By construction. |
| 2. $\angle X = \angle XCA$ | 2. By construction. |
| 3. $\therefore AX = AC$ | 3. If the base angles of a $\Delta$ are equal, the $\Delta$ is isosceles. |
| 4. $\therefore AX - AB = AC - AB$ | 4. Equals from equals give equals. |
| 5. $XB = AC - AB$ | 5. By substitution. |
| 6. But $XB = b - c$ | 6. By construction. |
| And | |
| 7. $\angle ABC =$ the given $\angle B$ | 7. By construction. |
| 8. $\therefore \Delta ABC$ is the required triangle | 8. It has the required parts. |

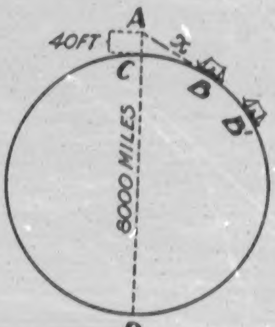
**Solving Problems.** Too often pupils solve problems by trying one thing after another, without much system. A teacher who is a master of the analytic method will train his pupils in regular analytic methods of thinking out the solution of problems. Problems of computation in geometry should always be solved by following a general plan, thus:

1. *Draw a diagram, if possible, marking on the figure the given data.*
2. *Represent the parts of the figure to be computed by the appropriate algebraic symbols.*
3. *Apply theorems to the figure, which give relations connecting the given and required parts, and derive equations from them.*
4. *Solve the resulting equations.*

**Illustrative Example.** *The eye of an observer at sea on the deck of a vessel is 40 ft. above the water line. The entire hull of a boat has just become visible on the horizon. Assuming the diameter of the earth to be 8000 miles, how far away from the observer is the boat?*

**Solution.** 1. We first draw a diagram representing the conditions of the problem.

In the diagram, $AC$ represents the elevation of the observer above the water line, at $C$. Since the hull of the boat has become entirely visible, it must have advanced from the position $B'$ to $B$, where the line of sight $AB$ is tangent to the surface of the water. $DC = 8000$ mi., $AC = 40$ ft.



2. The length of $AB$ (at sea, $AB = CB$ practically) is to be found. Let us represent it by $x$.

3. (a) We can solve for $x$ if we can bring $x$ into an equation connecting it with the given quantities, $AC$ and $CD$.

(b) To do this, we must express $AC$ and $CD$ in terms of the same unit of measure. Therefore, we select a mile as the unit. $AC = 40$ ft. $= 0.0072$ mi., and $CD = 8{,}000$ mi.

(c) In the diagram, $AD$ is a secant, $AC$ is its external segment and $AB$ is a tangent from $A$. Therefore, we can bring $x$ into a proportion (equation), with $AD$ and $AC$, both known quantities:

$$\frac{AD}{x} = \frac{x}{AC}$$

or

$$\frac{8000.0072}{x} = \frac{x}{0.0072}$$

4. Since in such a proportion the product of the means is equal to the product of the extremes, we have:

$$x^2 = 0.0072 \times 8000.0072$$

and $x = 57.6$ mi. approximately.

It is easy to get the student to see that in his solution the decimal 0.0072 added to 8000 does not appreciably affect the product 57.6. He sees that practically the same result is obtained if we write

$$8000 : x = x : 0.0072.$$

He is now in a position to solve the PROBLEM:

*"Derive a formula for the distance in miles to the visible horizon, if the eye of the observer is h ft. above the earth's surface."*

He easily writes

$$\frac{8000}{x} = \frac{x}{\frac{h}{5280}}$$

and finds by solving for $x$ that

$$x = \sqrt{1.54\,h} \text{ approx.},$$

$$\text{or roughly, } x = \sqrt{3/2\,h}.$$

Students trained to think analytically do not waste time in fruitless attempts, false starts, and discouraging and worthless computation. The solution proceeds in most cases with the certainty and ease of a logical machine.

### III. Conclusions for Curriculum Making in Geometry

**Emphasize the Analytic Method.** The first conclusion that emerges from the above brief survey seems to be that if the analytic method is the heart of logical training in geometry, sufficient time must be given to the presentation of the subject to enable the student to master this method of organizing the subject matter. Six or eight weeks devoted to committing to memory from twelve to eighteen theorems with their demonstrations will not give the student any insight into the true nature of geometric thinking. It would be much better to cover the first three theorems of congruence and the theorem about the base angles of an isosceles triangle, *with numerous original exercises*, developed analytically, than to have the student memorize any number of demonstrations. Of course, after the analysis, the synthetic demonstration should be given in every case, oral and written forms being emphasized.

**Summary of Experience with the Analytic Method.** Experience in teaching geometry by the analytic method and study of the results of numerous tests of the students' ability to analyze, extending over a period of more than twenty years in the High School of Commerce, New York City, have convinced me of the following facts:

1. A comprehension of analysis and some mastery of the system begin to emerge in the minds of the brighter pupils after six or eight weeks' instruction and practice in the method. The class as a whole begins to feel sure of the method after about ten weeks. Adequate mastery with accompanying pleasure and a thirst for

original exercises comes in the second semester. A school year should be devoted to the subject.

2. The student needs training in analysis applied to the various types of geometric subject matter found in the conventional five books of plane geometry. This need not be given in the order conventionally followed, but all the types should be included: congruence, similarity and proportion, circles, areas, constructions, and loci all call for analytic thinking, and help to clarify the method.

3. Teaching the method of analysis is difficult in the early stages. After it is begun, it should be continued until a clear concept of the method begins to emerge in the minds of the pupils. My experience, observation, and study convince me that at least *twenty weeks* of consecutive work in the subject matter should be given without a break of serious length. If a break comes then, it should mean the introduction of subject matter which is closely allied and which permits a continuation of the analytic method of attack. My own conviction is that *a year's consecutive work produces the best results* with the type of students entering our high schools. The continuous exercise in analysis and synthetic demonstration thus acquired leads to such a mastery that the students generally find a joy in geometric thinking.

4. To master plane geometry, using the analytic method of attack, requires about a year for the majority of pupils. If an introduction to demonstrative geometry is given in a regular junior high school, in which the analytic method has been used, then *the remainder of plane geometry* as well as solid geometry can be covered successfully in *one year* in the senior high school.

**A One-Year Course in Plane and Solid Geometry.** If, however, the attempt is made to *begin* plane geometry and teach both plane and solid geometry in one year, all in the senior high school, the following results are almost certain to appear:

(a) The amount of subject matter in both plane and solid geometry will have to be cut to such an extent that the student's view of both sciences becomes inadequate. He will not have a feeling of mastery of either at the end of the year's work.

(b) Because of the eagerness of teachers to cover a fair portion of both plane and solid geometry, a representative selection of book propositions with their proofs will probably be presented to the classes, the proofs committed to memory, and the chief value of the logical training sacrificed. It takes time to develop analytic thinking, and under the urge of covering ground quickly the analytic method stands small chance of being used.

The difficulties of visualizing the figures of solid geometry are greater for the ordinary student than are those of plane geometry. Under the year plan for both, instead of acquiring confidence and mastery in the plane geometry field, the student is almost certain to meet discouragement because of the added difficulties of the solid geometry concepts and the rapid pace necessary to cover the ground. For the analytic method is not mastered except by analyzing numerous originals covering the various types of geometric subject matter. Sufficient variety is offered by plane geometry for this purpose, and a thorough mastery of the analytic method in this field takes about a year's time.

**Preserve the Analytic Method.** Whatever is done in rearranging the subject matter of geometry, we must hold fast to the analytic method of attack if we would preserve its value as a training in logic and in original thinking, and inspire the confidence which lies back of the bravery necessary to attack its difficulties.

> Only the brave may look on Beauty's face,
> Search out her secrets, stand before her there
> In temple vast, of number, time, and space.
> Austere and cold, she guards her treasures fair,
> Flashing a blinding light upon the race
> Of rash, heroic creatures of an hour,
> Searching infinity, whose dazzling haze
> Confounds Philosophy with Beauty's power.
> Only the brave will wander far by choice
> In Euclid's realm, and in that wondrous maze
> Of new relations hear that thrilling voice
> Proclaim the reign of law, necessity;
> E'en here perfection, lost each petty choice,
> Surrendered in the law's great majesty.

# SYMMETRY

BY JOHN W. YOUNG

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**What is Symmetry?** Symmetric forms abound in nature and in art. If we confine ourselves to bilateral symmetry, i.e., symmetry with respect to a line (axis–axial symmetry) or with respect to a plane, we find it exemplified in the external form of the human body and in that of most animals, in the shape of leaves and, approximately at least, in the growth of most plants; we see it in the construction of most articles of furniture, in many buildings, in parts of buildings such as windows, doorways, arches; we observe it in the designs of wall paper, rugs, linoleum, and the shapes of ornaments. The child grows up with symmetric forms all about him, even though he may not know the word "symmetry," and may find it difficult to give a precise definition of it. He will approach his first study of geometry with the idea of symmetry already present in his mental equipment.

It is for this reason that European schools have for some time made use of this idea in the introductory work in geometry. The fact that axial symmetry can be used as a tool, as a method of proof, in plane geometry does not appear to be so well known in this country. This must be the justification for the few pages that follow. They contain nothing in the slightest degree original. They will, it is hoped, offer something of interest to those teachers who have not as yet thought along these lines.

**Axial Symmetry.** If a plane be rotated about one of its lines as an axis through an angle of 180°, we will say that it is *turned over about the line.* We then define axial symmetry in the following way:

If a plane figure is such that, if it be turned over about a certain one of its lines, the new position of the figure coincides with the original, it is said to be *symmetric with respect to the line.* The line is called the *axis of symmetry.*

The axis evidently divides the figure into two halves each of which comes to coincide with the other when it is turned over about the axis. Two points $A$, $A'$ of the figure (Fig. 1) which are simply interchanged by turning the figure over about the axis $XY$ are said to be *corresponding points* of the figure. Every point of the axis corresponds to itself.

**Results from the Definition.** Certain results follow immediately from the definition:

1. *The axis of symmetry bisects the segment joining any two corresponding points and is perpendicular to it.*

For, by turning the figure over about the axis, it is seen that the two segments $MA$ and $MA'$ are equal, and the two supplementary angles $YMA$ and $YMA'$ are also equal.

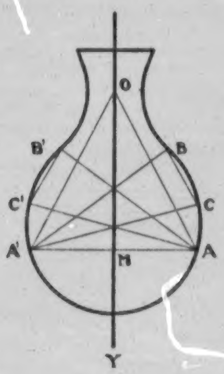

FIGURE 1

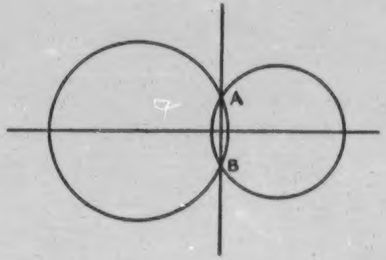

FIGURE 2

More generally we have:

2. *Any part of a symmetric figure is congruent to its corresponding part.*

In particular, the segment joining any two points of a symmetric figure is congruent to the segment joining the corresponding points; any angle or triangle $A'BC$ determined by three points is congruent to the angle or triangle determined by the corresponding points $AB'C'$. As a special case we note that the segment $OA$ joining any point $O$ of the axis to any point $A$ of the figure is equal to the segment joining $O$ to the corresponding point $A'$, and these two segments $OA$, $OA'$ make equal angles with the axis.

The reader should note that we draw the above conclusions without any previous knowledge of geometric theorems. We assume only a knowledge of the meaning of point, of straight line, of angle, of right angle, of perpendicular, of equal or congruent (by superposition), and of the fact that two distinct points determine a straight line. Such considerations as the above, stripped perhaps of some of their formality, may therefore be presented to a class very near the outset of its study of geometry. They would indeed seem to have their chief pedagogical value during the introductory parts of the subject.

We may now note the converse of 1:

3. *The perpendicular bisector of a line segment is an axis of symmetry of the segment.*

This gives us at once:

4. *Any point on the perpendicular bisector of a line segment is equidistant from the extremities of the segment; and the lines joining any such point to the extremities of the segment make equal angles with the bisector.* The reader will note further that certain fundamental theorems on isosceles triangles also follow at once.

The further use of symmetry as a method of proof depends on the following fundamental proposition:

5 *A circle is symmetric with respect to any straight line through its center*

Hence, we have:

6. *The figure formed by two circles is symmetric with respect to the line joining their centers. If the circles intersect, the points of intersection are corresponding points of the figure.* (Fig. 2)

This gives us:

7. *The common chord of two circles is perpendicular to the line joining their centers and is bisected by it.* (Fig. 2)

**Two Examples.** We will close this brief sketch by noting how two of the important elementary constructions may be justified by considerations of symmetry.

1. *To draw the perpendicular bisector of a segment.*

By what precedes, the problem will be solved if we can draw two circles intersecting in the extremities $A$ and $B$ of the segment. For then the line joining the centers of these circles will be the required bisector. (See Fig. 2, above) To this end, then, with $A$ and $B$ as centers and with any convenient radius let us describe two arcs of circles intersecting at $M$, $N$ (Fig. 3). Circles with centers at $M$ and $N$ and the same radius will then pass through $A$ and $B$. Hence $MN$ is the required bisector.

2. *To bisect a given angle.*

The usual construction consists in drawing a circular arc with center at $B$, the vertex of the given angle $ABC$ cutting the sides in $D$ and $E$, respectively; then, with any convenient radius and with centers at $D$ and $E$, describing arcs intersecting in $M$ (Fig. 4). The line joining the vertex $B$ to $M$ is the required bisector. For we have a circle with center $B$ passing through $D$ and $E$, and the construction shows that there is a second circle with center at $M$ passing through $D$ and $E$. The figure is then symmetric with respect to the line of centers $BM$, and thus $BM$ bisects the angle $ABC$.

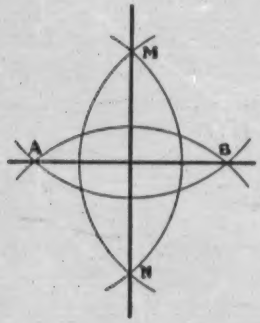

FIGURE 3

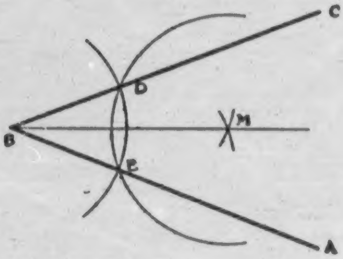

FIGURE 4

Enough has been said, it is hoped, to indicate how the idea of symmetry can be *used* in elementary instruction in geometry. Other possibilities will suggest themselves to the progressive teacher. It may be expected that the work in intuitive geometry especially can be benefited by the use of symmetry.

# THE TRANSFER OF TRAINING, WITH PARTICULAR REFERENCE TO GEOMETRY *

By WILLIAM BETZ

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**Foreword.** Demonstrative geometry, as everyone knows, has become a highly controversial subject. Once regarded as the road *par excellence* to all scientific and philosophical thinking (Plato), it is now denounced by a chorus of self-appointed critics as a species of Greek philosophy that should be eliminated from the crowded curricula of our secondary schools. Geometry shares with algebra and Latin the fate of being on the defensive. Its enemies claim that as a school subject it is kept alive artificially either by the requirements of blind tradition or by obviously false pretenses that are said to have their roots in the "exploded" theory of mental discipline. The *practical* phases of the subject, it is asserted, could be covered in a few lessons or taken care of incidentally in the intuitive geometry course. And as to the alleged cultural and disciplinary values that have always been associated with demonstrative geometry, there is no disposition on the part of the critics even to argue the question in the light of expert testimony or of a scientific exposition of the real nature of geometric training.

This condition of affairs has been accentuated by the revolutionary educational changes and the numerous transforming influences of the past generation. The spectacular increase in the enrollment of our high schools has produced difficult administrative and pedagogic problems which thus far have been solved but imperfectly or not at all. Teachers have been put to it to "justify" every lesson they teach. The endeavor to protect the child against "unessentials" is rapidly leading to curious and unsuspected consequences. Geometry, in particular, is a popular target of abuse. Being rated as a "high mortality" subject, its position in the program of studies is becoming increasingly delicate and precarious.

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* This monograph was prepared in connection with the curriculum revision program of the elementary and secondary schools of Rochester, N. Y.

How much justice is there in the wave of condemnation that appears to be engulfing so time-honored a subject?

Unfortunately, the teachers of geometry, who should have been its most enthusiastic and successful exponents, have only too often been its worst enemies by their lack of acquaintance with its history and its distinctive characteristics, and by their apparent inability to formulate and to realize the immediate and the ultimate objectives of the subject.

To make matters worse, our textbooks, courses of study, and examinations convey hardly a hint of the astounding scientific transformation which mathematics, including geometry, has been experiencing during the past century. Thus the arrival of non-Euclidean geometric systems, the creation of projective geometry and of hyper-spaces, has made it clear that it is impossible, as Gauss suspected, to state with finality an exclusive body of truths concerning "absolute space." What we can do is merely to proceed from certain significant spatial assumptions to logical consequences of these assumptions. We can merely say—"If *this* is true, then *that* is true."

It would seem at first that with the disappearance of the absoluteness of Euclid's system of propositions, its chief claim to serious consideration as a basic school subject had vanished. There are, however, two crucial reasons why demonstrative geometry will remain for all time a necessary subject of instruction.

In the first place, we do not stop teaching any one of the physical sciences when it is discovered that certain of its underlying hypotheses must be modified. Scientists have been quarreling over the nature of light, the mystery of gravitation, the constitution of matter or of electricity, and the like, for many years. And yet we go on teaching each generation of pupils "the basic scientific facts" as we understand them. There is nothing dishonest in this procedure, provided we refrain from making dogmatic assertions instead of expounding provisional hypotheses. In like manner, since space is a permanent category of our thinking, we shall always have to study and formulate, to the best of our ability, the spatial truths that seem to account best for the segment of reality with which we can deal. And *Euclid's system thus far has proved to be the simplest and most convenient one for everyday use.*

But there is a very much deeper reason for a continued emphasis on geometry in the curricula of our schools. It has been set forth recently in almost classic fashion by Professor C. J. Keyser.[1] Euclid's *Elements of Geometry*, though by no means without flaws, was the first instance in human history of autonomous thinking. In this book the human mind gave its first evidence of complete scientific awakeness, for in it we find the earliest example of the use of a postulational system for purposes of careful deduction. As Professor Keyser points out, this colossal achievement remained unique for many centuries, but it has since become the prototype of *all* valid scientific investigations. And thus geometry has given to the world its one reliable method of thinking. *Postulational thinking* is now seen to be of basic importance in any field of human endeavor. Even a cursory analysis shows that *demonstrative geometry offers the simplest and most convenient introduction to postulational thinking which has yet been devised.* Hence it may be claimed that the teaching of demonstrative geometry is not only justifiable, but absolutely essential, because of its permanent devotion, in a singularly pure and significant form, to the one procedure that promises valid conclusions on the basis of clearly formulated assumptions.

The acceptance of this thesis is seen to depend, however, on the psychological question whether training in *postulational* thinking, in *critical* thinking, and in *consecutive* thinking, when administered in geometric form, can be made available in *other* fields of work. That is, *how far is geometric training capable of "transfer"?*

Obviously, this question is of fundamental importance. Only a reassuring answer will rescue geometry as a required school subject. Such an answer can no longer be based entirely on subjective opinions. Instead, it presupposes a painstaking scientific examination.

It is the purpose of this chapter to offer a brief account of the present status of the problem of transfer. A comprehensive review of this question would naturally have to include a large number of highly *technical* considerations. In this report we are primarily concerned, however, with the *practical* aspects of the problem of transfer, especially in so far as these affect the teaching of geometry.

The opinions of outstanding psychologists and educational spe-

[1] See, especially, the following three treatises by Professor Keyser: *Thinking About Thinking*, E. P. Dutton and Co., 1926; *The Human Worth of Rigorous Thinking*, Columbia University Press, 1916; and *Mathematical Philosophy*, E. P. Dutton and Co., 1922.

cialists will be quoted at length, in order that the reader may have a more objective basis for an independent appraisal of the conclusions that will be submitted. The free use of italics, even when not suggested by the original text of the quotations, seems desirable for the sake of emphasis. After a brief summary of the progress of the mental discipline controversy, followed by a review of certain assumptions and definitions which underlie the testing movement, we shall examine the experimental evidence as to *the extent of transfer.* It will then be necessary to analyze the principal theories as to *the mode of transfer.* From this dual foundation we shall be able to derive the pedagogic consequences in which we are primarily interested.

It will be found, at the end of our discussion, that there is excellent reason for an optimistic attitude with reference to the reality of "mental discipline," and—in particular—for a continued belief in the permanent cultural significance of geometric training.

## I. THE MENTAL DISCIPLINE CONTROVERSY

**Nature and Importance of the Controversy.** During the past four decades no question has occupied a more prominent place in educational literature than the one which we are to study in this chapter. For a long time it was known as the problem of "mental discipline" or of "formal training." More recently, it is being referred to as the "transfer" or "spread" of training, or as the problem of "generalized experience."

As originally advocated, the dogma of mental discipline asserted that *the formal training or the mental power gained from the study of certain school subjects carries over to all other activities.* "No educational theory has ever exerted the profound influence upon curriculum-making and methods of teaching as this doctrine has done."[3]

Believers in this doctrine maintained that "the chief, if not the sole, value of the educative process consists in the formal development of the mind's powers, in producing a fund of mental force or strength, and in establishing certain generalized habits. Content or intrinsic values are either disregarded altogether, or are given a secondary position. It makes little difference what is studied so long as it is studied right. The benefit received comes from the process of acquisition rather than from the content acquired. The powers and habits once developed may then be applied in the various activities of life with little or no loss of effect. The power of reasoning developed in mathematics or logic may later be used in law, medicine, or business; and the habit of concentration developed in solving problems in cube root or in translating Greek may be likewise extended. Observation, memory, diligence, accuracy, and other habits and powers are taken to be subject to the same rule . . . From this point of view the mind may be likened to a storage battery that may be charged, and the power accumulated may then be used quite independent of its origin."[3]

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[3] See Turner, E. M. and Betts, G. H., *Laboratory Studies in Educational Psychology,* pp. 174-176. D. Appleton and Co., 1924.

The following more recent formulation shows a tendency toward a cautious limitation of these sweeping claims:

> The problem may be stated in the following words: Will the formation of one habit either help or hinder in the formation of another? Will the acquiring of one bit of knowledge help or hinder in acquiring other knowledge? . . . does the learning of one language make the learning of a different language easier? Will the study of history make the acquisition of mathematics easier? In general, are the results of learning narrow and specific, or are there general effects also?[4]

Again, a reinterpretation of the problem in the direction of "generalized training" is reflected in these interesting queries:

> What would be the effect of four years' earnest effort to excel in classical languages and mathematics upon one's ability to master the intricacies of banking, or upon one's persistence and doggedness in the face of any other complex problem? What is the effect of four years' work in the high school and four years in the university upon the probability that one will continue to master new problems afterwards—that is, upon the probability that life will be a career instead of merely the holding down of a job? What would be the effect of writing a first class doctor's dissertation, say on education, upon one's ability to organize the advertising department of a great industry? To what extent will prolonged and intensive mental effort tend to inure one to the onerousness of mental effort? Does extensive experimentation in finding solutions of difficult problems tend to make one more enterprising and persistent in casting about for the solutions of other problems?[5]

There can be no doubt that "the problem raised here is of far-reaching theoretical and practical significance. It involves the very foundation principles of education, and we must face it at every turn in practice." To quote Pyle further:

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[3] Ruediger, W. C., *The Principles of Education*, pp. 91-92. Houghton Mifflin Co., 1910.

[4] Pyle, W. H., *The Psychology of Learning*, p. 213. Warwick and York, Inc., 1921.

[5] Lennes, N. J., "A Mathematician on the Present Status of the Formal Discipline Controversy." *School and Society*, Vol. XVII, p. 69, January, 1923.

> It involves our fundamental conception of the nature of mind. . . . Our answer to these questions will color our whole scheme of education. If the results of training are *specific*, then we should learn those things which we most need to know, without any reference to their general effects. If the results of training are *general*, then we should pick out as the studies for our curriculum, those branches which are best for the exercise of the mind.[6]

**Development of the Controversy.** The doctrine of formal discipline has been held, in one form or another, throughout educational history. Its modern form may be traced to John Locke (1623-1704). Thus, in his "Conduct of the Understanding" we find passages such as the following:

> Would you have a man reason well, you must use him to it betimes; exercise his mind in observing the connection of ideas and following them in train. Nothing does this better than mathematics, which therefore should be taught all those who have the time and opportunity, not so much to make them mathematicians as to make them reasonable creatures. . . . Not that I think it necessary that all men should be deep mathematicians, but that, having got the way of reasoning, which that study necessarily brings the mind to, they might be able to transfer it to other parts of knowledge as they shall have occasion.[7]

Similar ideas were widely prevalent, for a long time, as might be proved by an extensive series of quotations from educational literature.[8] A single illustration must suffice. Says Joseph Payne:

> The study of the Latin language itself does eminently discipline the faculties and secure to a greater degree than that of the other subjects we have discussed, the formation and growth of those mental qualities which are *the best preparatives for the business of life*. . . .

The reaction against these extreme views began in Germany among the Herbartians, especially through the writings of Tuiskon Ziller. Stimulated by Ziller's criticisms, Dr. Elmer Ellsworth Brown, former United States Commissioner of Education, published the first critical discussion in America, in a paper entitled "How is Formal Culture Possible?" in the *Public School Journal* for December, 1893. The first paper, however, which succeeded in drawing

[6] Pyle, *op. cit.*, p. 213.

[7] Ruediger, *op. cit.*, pp. 77-78.

[8] For a list of such quotations see Ruediger, *op. cit.*, pp. 86-90; also Thorndike, E. L., *Educational Psychology, Briefer Course*, pp. 269-272, Teachers College, Columbia University, 1919.

the attention of American educators to this question was that of Hinsdale, on "The Dogma of Formal Discipline," which was read before the 1894 meeting of the National Education Association at Asbury Park, New Jersey. The doctrine has been under discussion ever since.[9] As a result, a vast literature on the subject has sprung up. Above all, the question has been attacked scientifically, so that we now have an imposing mass of objective laboratory data instead of an array of unsupported opinions.[10]

The arrival of the measurement movement greatly stimulated interest in the question. It furnished important techniques of investigation and showed the need of greater moderation with reference to the alleged reality of formal training. For a time, it looked as if all belief in "general" discipline had to be abandoned, and as if only "specific" training resulted from any type of learning.

To-day, "neither the extreme view supporting the theory of mental discipline in all respects and under all circumstances, nor the opposite view rejecting it completely and utterly, is in good repute. Instead, a compromise theory—an intermediate judgment —holds sway."[11] This newer belief still maintains that mental discipline and intellectual and moral powers "are the very ends for which education exists," but it does not expect the attainment of this broad training to result either automatically or from the mechanical cultivation of a few highly favored activities.

## II. Theories of the Mind and Basic Definitions

**Permanent Educational Problems.** Every human being is born into a universe of apparently infinite dimensions, of inconceivably complex relationships and correspondingly varied possibilities. The curriculum is the principal means of interpreting the world of nature and of man to this frail and limited being. Viewed in this light, the school faces a program of overwhelming magnitude. Of necessity, the curriculum can never offer more than a very modest cross section of all possible experience; and yet this cross section—however fragmentary—is the pupil's primary tool for educational progress and potential growth. Hence the constant debate about educational values, about objectives and the organization of materials of instruction; hence also the perennially important problem of explaining how a limited human mind can cope with the inexhaustible educational resources which modern society has accumulated.

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[9] For a brief account of the mental discipline controversy in America, see Ruediger, *op. cit.*, pp. 96-101.

[10] An extensive bibliography of the recent literature on the transfer of training is given in Mead, Arthur Raymond, *Learning and Teaching*, pp. 118-122. J. B. Lippincott Co., 1923. See also Whipple, Guy M., "The Transfer of Training," in *The Twenty-seventh Yearbook of the National Society for the Study of Education*, Part II, pp. 179-209.

[11] Davis, C. O., *Our Evolving High School Curriculum*, p. 132. World Book Co., 1927.

**Theories of the Mind.**[12] The older doctrine of mental discipline was based on an erroneous "faculty psychology." It assumed that the mind was composed of various compartments, each of which functioned in its entirety and could therefore be made the object of intensive training by any pertinent activity. This simple but naïve and unscientific view was held almost universally until quite recently. It had to be abandoned when experimental psychology showed unmistakably that any one of these alleged "faculties" in reality represents a very diversified domain. Thus it was found that a person may have a good memory for words, but a poor memory for spatial forms. Discoveries of this type led to a violent reaction against the older view. Instead, the doctrine of extreme specialization of mental functions became popular among psychologists. It became the fashion, especially among those who were uncritical or merely uninformed, to speak contemptuously of the "exploded myth of mental discipline." The result was an almost hopeless confusion of ideas. For, if all learning can be shown to be strictly specific, the curriculum must abandon all hope of securing worth-while educational results on the basis of the "spread" or "carry-over" of broad cultural fields of work. Such a conclusion seemed utterly at variance, however, with common sense and with the experience of the ages.

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[12] For obvious reasons, the perennial debate among psychologists and philosophers as to the nature of soul, mind, intelligence, consciousness, meaning, reality, value, and a host of related problems, can only be alluded to in these pages. It is certainly true that "psychology at present is a scene of confusion and violent disagreement. There is a steadily mounting mass of data, but we do not know what they mean." It has even been said that at present "there is no such thing as psychology. There are only psychologies." Generally speaking, we now have as many psychologies of learning as there are different theories of the mind. It is a matter of great importance which of these theories the teacher adopts. Each of them leads to a distinct conception of the learning process and of the problem of transfer which we are considering in this chapter. Hence any "conclusions" which may be submitted in a study like this will appeal only to those who accept the author's fundamental assumptions. Nevertheless, the prevailing divergencies of opinion need not prevent us from agreeing that "the question of mind is of central importance, both for teaching method and for our whole program of education." (See Bode, B. H., *Conflicting Psychologies of Learning*. D. C. Heath and Co., 1929.)

The two theories of the mind referred to above may be pictured by diagrams such as the following:

Theory of the Faculty Psychology

| | | | | | |
|---|---|---|---|---|---|
| Perception | Thought | Will | Memory | Emotion | ...... |

Theory of Specialized Mental Functions

| | | | | | |
|---|---|---|---|---|---|
| Memory | | | | | |
| Words | Number | Color | Sound | Forms | ...... |

It is now known that neither theory states the case completely. A compromise theory is seen to be necessary, if we would account adequately for observed facts of transfer. As Colvin put it:

> The faculty psychology assumed a number of fabulous entities which worked out the destinies of the individual, while the doctrine of absolute localization of nervous function has made the brain a machine of relatively unrelated parts and has created a doctrine of psychic atomism which is as untrue as it is impossible of practical application. . . . In short, if we try to overthrow the doctrine of transfer on the ground of absolute localization of nervous functions, we are doing so on dubious theoretical grounds, and holding to a theory which runs counter to what we know of mental elements and mental organization. If, on the other hand, we accept the doctrine of *relative* rather than *absolute localization*, of colligation of remote functional areas, and of vicarious functioning (as does Wundt), we find that such an hypothesis, instead of making against the possibility of transfer, gives a clear basis and reason for such transfer. Indeed, a rational hypothesis of cerebral localization suggests coöperation and transfer of the widest possible sort.[illegible]

**The Mechanism of Thinking and Learning.** Any adequate discussion of "mental training," "transfer," and the like, presupposes a tentative agreement as to the meaning of certain psychological terms such as "learning," "mental functions," "efficiency," "improvement," and so on. For detailed and comprehensive definitions of these terms the reader must be referred to authoritative scientific treatises on the psychology of learning. Certain fundamental explanations are, however, essential at this point.

The brain, universally regarded as the organ of thinking, is now known to be an instrument of stupendous complexity. This fact

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[illegible] Colvin, S. S., *The Learning Process*, pp. 212, 237. Macmillan Co., 1917.

has recently been stressed by Professor C. H. Judd in the following manner:

> There is nothing more complex or more highly integrated than human thinking. There is nothing more original in the world than human combinations of ideas. . . . In the case of man there has appeared as the essential fact in his bodily equipment a larger and more highly organized cerebrum than that possessed by any other animal. In this cerebrum sensory and motor impulses unite in associative combinations. *In every normal human being there is an inner world of ideas and of recognitions of values, for which inner world of rational thought there is no counterpart in the world studied by the physicist or in life below the human level.* . . . He uses a cerebrum in which the associative processes, which combine and recombine nervous impulses, are the typical and significant facts in his life. . . . *We must conceive of the inner processes of reasoning as related to elaborate systems of organized tracts in the cerebrum similar to those involved in speech.* It is through the functioning of such cerebral tracts that man has been able to achieve through mechanical invention and scientific thought the supreme place which he occupies in the world. It is in the cerebrum, rather than in the lower reflex and automatic centers, that the new combinations of sensory and motor impulses have been worked out which give to man the degree of mastery of his environment which he has thus far achieved.[14]

It is with the aid of this elaborate cerebral equipment that all higher forms of learning take place.[15] But what is *"learning"?* Without committing ourselves in every detail to a "stimulus-response" psychology of learning, we may yet accept for our present purpose—in a purely provisional way—Professor Thorndike's well-known definition:

> *Learning is connecting,* and man is the great learner primarily because he forms so many *connections.* The processes described in the last two chapters, operating in a man of average capacity to learn, and under the conditions of modern civilized life, soon change the man into a wonderfully elaborate and intricate *system of connections.* There are millions of them. They include connections with subtle abstract elements or aspects or constituents of things and events, as well as with the concrete things and events themselves. . . .
>
> Any one thing or element has many different *bonds.* . . . Of the connections to be studied in man's learning an enormous majority begin and end with some state of affairs within the man's own brain—are *bonds between one*

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[14] See *The Nature of the World and of Man,* H. H. Newman, Editor, Chapter XVII, "Mind in Evolution," pp. 527 ff. University of Chicago Press, 1926.

[15] The most determined recent attempt to describe the processes of thinking in the light of modern neurological research is represented by the following three volumes written by Professor C. J. Herrick: *An Introduction to Neurology,* W. B. Saunders Company, 1922; *Brains of Rats and Men,* University of Chicago Press, 1926; *The Thinking Machine,* University of Chicago Press, 1929.

*mental fact and another.* The laws whereby these connections are made are significant for education and all other branches of human engineering. *Learning is connecting;* and teaching is the arrangement of situations which will lead to desirable bonds and make them satisfying.[16]

**Mental Functions and Related Terms.** In the light of the broad definition of "learning" suggested above we may now turn to related psychological terms which are of importance in a scientific study of the problem of transfer of training.

*Mental Functions.* "Let us use the term *mental function* for any group of connections, or for any feature of any group of connections, or for any segment or feature of behavior, which any competent student has chosen or may in the future choose to study, as a part of the total which we call a man's intellect, character, skill, and temperament. By so catholic a definition we shall have a convenient term to mean *any learnable thing in man,* the psychology of whose learning anybody has investigated. . . . Mental functions may be 'wide' or 'narrow'. . . . A mental function may involve a single set, or a series of sets, of bonds—may be 'short' or 'long'. . . . A mental function may relate primarily to the form of what is done, or to the content in connection with which something is done. . . . A mental function may consist primarily in an attitude or primarily in an ability." [17]

*Intelligence.* "The term *'general intelligence'* should be provisionally accepted as connoting the most important function of mind,—namely the ability to control behavior in the light of experience." [18]

*Efficiency.* We may say that "*the efficiency of a mental function* is the status of that function at any given time with regard to its quantity and quality. For example, a pupil can get sums of four four-place numbers to-day at the rate of 2 per minute and 65 percent of his sums are accurate. 'Two per minute' is the *quantity;* '65 percent accurate' is the *quality.*" [19]

*Improvement or Deterioration.* "*Improvement* of a mental function is an increase in its quantity, its quality, or increase in both. For example, the pupil referred to above, after a week's practice, had the following efficiency: 3 per minute and 75 percent accurate. The change was '1 per minute' and '10 percent accurate'. Both constitute improvement. Likewise, *deterioration* is a *decrease* in quantity, quality, or both. Suppose that the pupil's efficiency after practice had been '1.8 per minute' and '40 percent accurate'. Then the loss would have been '0.2 per minute' and '25 percent accurate'. These two losses would constitute *deterioration.*" [20]

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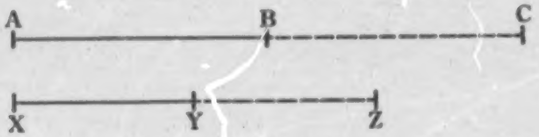
[16] Thorndike, *op. cit.*, pp. 173-174.

[17] *Ibid.*, pp. 176 ff.

[18] Bagley, W. C., *Determinism in Education*, p. 157. Warwick and York, 1925.

[19] Mead, *op. cit.*, p. 94.

**Definition of "Transfer."** Let *AB*, in the diagram, represent a pupil's equipment, say in mathematics, at the beginning of a certain period, while *BC* is his *improvement* in that subject after a period of much drill. Also, let *XY* represent a collection of other mental functions, while *YZ* is the supposed improvement in *XY due to the improvement BC in AB.* This improvement *YZ* is the real or alleged "general mental training" resulting from the activities which caused the improvement in *AB.*[21]



Transfer of training may therefore be defined as "*the influence which an improvement or change in one mental function has upon other mental functions.*" [22]

The relation of *BC* and *YZ* in the diagram above, formerly a matter of dogmatic assertion, is now seen to be a problem requiring careful investigation. We must therefore turn to the evidence for or against "transfer" which experimental psychology has accumulated by the application of "scientific" methods.

## III. THE EXPERIMENTAL INVESTIGATION OF TRANSFER

**Possible "Transfer" Relations of Mental Functions.** Suppose that *A* and *B* are two mental functions which are to be investigated with regard to transfer, *A* being the function improved by training, "*the influencing function,*" while *B* is the untrained function, the "*function to be influenced.*" We may then have the following possible cases:

Case I. Improvement in *A* improves *B*.
Case II. Deterioration in *A* weakens *B*.

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[20] Mead, *op. cit.*, p. 94.
[21] *Ibid.*, pp. 92-93.
[22] Thorndike, *op. cit.*, p. 259.

Case III. Improvement in *A* weakens *B*.
Case IV. Deterioration in *A* improves *B*.
Case V. Either improvement or deterioration in *A* is of no influence on *B*.[23]

These relations may be summarized by a diagram such as the following:

|  | I | II | III | IV | V |
|---|---|---|---|---|---|
| *A* | + | – | + | – | ± |
| *B* | + | – | – | + | 0 |

The literature of experimental psychology abounds in illustrations of these various cases.[24] We are mainly interested, however, in the question of *the relation of improvement in one mental function to an increased efficiency of other functions.* That is, Case I is the primary object of our discussion. We shall therefore proceed to examine the technique by means of which an improvement in a certain function *A* and its *resulting* relation to a second function *B* may be investigated.

**Method of Investigation.** The typical procedure now followed in the experimental determination of transfer involves the existence of two groups of "subjects," say high school pupils, which must be as nearly as possible of equal ability. The selection of these groups is based on careful preliminary intelligence tests. Group I is called the "practice" group, while Group II is the "control" group. Both groups are then given an initial test in the abilities to be investigated, say in *A* and *B*, thus determining their respective "base lines" in these abilities. Then Group I is given special practice in ability A, while Group II is *not* given such practice. At the end of the training period both groups are again tested in the same abilities. The scores of each group in these final tests are noted. If Group I scores a greater relative gain in *B* than Group II, all other things being equal, this superiority may be attributed to the special training enjoyed by Group I.

The necessity of a "control" group was not realized in the earliest experiments. It is now seen to be of crucial importance, as the following statement suggests:

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[23] See Mead, *op. cit.*, pp. 97 ff.

[24] Thus, for a discussion of *negative* transfer, see Judd, C. H., *Psychology of Secondary Education*, p. 430. Ginn and Co., 1927.

Suppose there are ten aspects of memory, and we wish to learn whether training in aspect Number 5 will improve all the other nine aspects. We take a group of people and measure all 10 aspects of memory; we then train the group in aspect Number 5 until there is great improvement and then measure again in the other nine aspects. Suppose we find that there is improvement in all the other nine aspects; we cannot say that it is due to the practice in Number 5, because, for all we know to the contrary, the group might have made this improvement *without* the practice. It may be that if we give the ten tests and then wait a few weeks and give ten similar tests again, there will be considerable improvement. In fact, such is usually the case. We must, therefore, in an experiment of this kind, take *two* groups and give one group the initial and final tests and give the other group the same initial and final tests and the *special practice besides.* Then, whatever differences in the final tests are not otherwise accounted for, may be considered to be due to the practice.[25]

**Debatable Assumptions Underlying the Technique of Mental Tests.** The investigation of transfer by means of mental tests of the usual type is subject to certain inherent weaknesses. It is only too true, as Inglis reminded us, that "the science of experimental psychology, in spite of its rapid and promising development within recent years, is still in its infancy. Hence the tools which experimental pedagogy must employ are as yet of the crudest."[26] In particular, one may have serious misgivings as to a purely *quantitative* evaluation of mental phenomena. With evident disregard of age-old philosophic speculations and difficulties, prominent exponents of the measurement movement in education do not hesitate to propose "theses" such as the following:

1. "Whatever exists at all, exists in some amount." (Thorndike)
2. "Anything that exists in amount can be measured."
3. "Measurement in education is in general the same as measurement in the physical sciences."[27]

Recently Thorndike and his colleagues of the Institute of Educational Research of Teachers College, Columbia University, issued an extensive report on *The Measurement of Intelligence,* in which this quantitative doctrine is reasserted with great confidence. Thus, Chapter XV contains the following significant passages:

[25] Pyle, *op. cit.*, p. 216.

[26] Inglis, A., *Principles of Secondary Education*, p. 408. Hodghton Mifflin Co., 1918.

[27] See McCall, W. A., *How to Measure in Education*, Chapter I. The Macmillan Co., 1922.

> The standard orthodox view of the *surface nature of intellect* has been that it is divided rather sharply into a *lower half, mere connection-forming or the association of ideas, which acquires information and specialized habits of thinking; and a higher half characterized by abstraction, generalization, the perception and use of relations and the selection and control of habits in inference or reasoning, and ability to manage novel or original tasks.* The orthodox view of its deeper nature, so far as this has received attention, has been that the mere connection or association of ideas depends upon the physiological mechanism whereby a nerve stimulus is conducted to and excites action in neurones A, B, C, rather than any others, but that the higher processes depend upon something quite different. There would be little agreement as to what this something was, indeed little effort to think or imagine what it could be, but there would be much confidence that it was *not* the mechanism of habit formation.
>
> *The hypothesis which we present and shall defend admits the distinction in respect of surface behavior, but asserts that in their deeper nature the higher forms of intellectual operation are identical with mere association or connection forming, depending upon the same sort of physiological connections but requiring many more of them.* By the same argument the person whose intellect is greater or higher or better than that of another person differs from him in the last analysis in having, not a new sort of physiological process, but *simply a larger number of connections of the ordinary sort.* . . .
>
> The essential element of our hypothesis is that it offers a *purely quantitative fact,* the number of c's [connections], as the cause of qualitative differences either in the kind of operation (e.g. association versus reasoning) or in the quality of the result obtained (e.g. truth versus error, wisdom versus folly), so far as these qualitative differences are caused by original nature. . . . We shall not discuss general arguments pro and con in this report, but will simply note that both the phylogeny and the ontogeny of intellect seem to us to show *selection, analysis, abstraction, generalization,* and *reasoning coming as a direct consequence of increase in the number of connections;* and that what little is known of the status of the neurones in very dull individuals is in harmony with the quantitative theory.[28]

Without commenting further at this point on the basic philosophic questions raised by such a purely mechanistic orientation,[29]

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[28] Thorndike, E. L., *The Measurement of Intelligence,* Chapter XV. Teachers College, Columbia University, 1926.

[29] See, for example, McDougall's keen analysis of the quarrel between the "mentalists" and the "mechanists," in his Clark University lectures on "Men or Robots," published in *Psychologies of 1925*, pp. 273-305, Clark University, 1926. Says Professor McDougall:

"Even if the faith of the mechanist were well grounded and justified, even if we had some impossible, some supernatural, assurance of this, it would still be more profitable now and for a long time to come to make use of the mentalist, the truly purposive, interpretation of behavior; *for we are very, very far from any adequate mechanical interpretations;* and we may hope to arrive at them most rapidly by continuing to use and to improve our mentalist interpretations, to formulate laws of behavior in mentalist terms, postponing the translation of them into terms of mechanism until such time as such interpretation may be a possibility and not let us turn to a brief survey of the literature dealing with the experimental determination of transfer.

**Experimental Studies of Transfer.** Even a brief sketch of the principal experimental studies of transfer would completely transcend the scope of this monograph. Besides, such summaries are now readily available. Thus the *Report of the National Committee on Mathematical Requirements* includes, in the form of supplementary plates prepared by Dr. Harold Rugg and Miss Vevia Blair, a synoptic review of the "experimental literature of mental discipline," to accompany Chapter IX of the Report, on "The Present Status of Disciplinary Values in Education." Plate I presents an outline of *nine studies dealing with memory*, beginning with the classical experiment of William James (1890), and ending with Sleight's well-known investigation (1911). Plate II epitomizes *ten studies dealing with sensory and perceptual data*, all issued between 1901 and 1914. Plate III refers to *fourteen studies dealing with associative-motor habits and special school activities*, reported between 1897 and 1920. Plate IV summarizes *ten investigations into the relations of various fields of intellectual activities*, as reported between 1901 and 1920. With the aid of the references given in the Report, the reader may personally acquaint himself with the exact details of each of these investigations. Brief accounts of many of these studies may also be found in the recent treatises on experimental psychology.[30]

Since the publication of these studies some very extensive investigations of special interest to high school teachers have been carried on. The monumental Classical Investigation, sponsored by the American Classical League, tested the validity of certain aims

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merely a misleading pretense, as at present it is. . . . It may be that the faith of the mechanist is altogether ill-based and illusory. To all appearance the life-processes of living things are fundamentally different from inorganic processes; and we have no guarantee, no adequate ground for believing, that this appearance is illusory. And, if we uncritically adopt this mechanistic faith, and under its influence elaborate a picture of the world in mechanistic terms, we inevitably arrive at an absurd position, as the history of thought abundantly shows; we find we have created a picture of the world which leaves out of the picture entirely that mental process, that purposive striving, that creative activity, which has produced the picture; our conscious striving to construct the picture, our conscious appreciation and understanding of it when constructed, remain outside it as something whose relation to the picture is entirely unintelligible. And so we have to start all over again, and strive to 'remould it nearer to the heart's desire,' *the desire to understand man's place in the universe.*" (*Op. cit.*, p. 303.)

[30] See, for example, Starch, D., *Educational Psychology*, pp. 193-211, Macmillan Co., 1920; Pyle, *op. cit.*, pp. 217-229; Gates, A. I., *Psychology for Students of Education*, pp. 255-264, Macmillan Co., 1925.

or objectives in the teaching of Latin, with particular reference to transfer effects in the field of English. Thousands of pupils were tested and noteworthy conclusions were obtained.[31] At about the same time an extensive study of "Mental Discipline in High School Subjects" was conducted by Professor E. L. Thorndike. His report was published in the *Journal of Educational Psychology*, Vol. XV (1924). Over eight thousand pupils were involved in the tests. Significant findings resulted.[32] More recently, a similar study was undertaken by the Institute of Educational Research of Teachers College, Columbia University, with the participation of Professor Thorndike, which took into account the earlier investigations and established important relationships between individual high school subjects, or groups of subjects.[33]

A review of these investigations impresses one with the gradual extension of the machinery which appears to be necessary to arrive at worth-while conclusions. The later studies involved thousands of children, required an expensive research organization, and depended on the application of highly technical statistical devices. Only the future can determine whether a further extension of such intricate methods is feasible or significant.

**Evaluation of the Experimental Investigations.** A summary of significant inferences which may be drawn from the results of all but the most recent of these studies is given in Chapter IX of the *Report of the National Committee on Mathematical Requirements*.[34] These inferences are based on the replies of twenty-four psychologists to whom a special questionnaire suggested by all the available experimental data had been submitted. We must limit ourselves to a list of the outstanding conclusions:

1. *The two extreme views for and against disciplinary values practically no longer exist. As the question now stands, as transfer of training, the psychologists quoted here almost unanimously agree that transfer does exist.*
2. A large majority agree that there is a possibility of negative transfer, and of zero transfer, caused by interference effects.
3. Very few if any experiments have shown the full amount of transfer between the fields chosen for investigation.

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[31] See General Report, Part I, of *The Classical Investigation*, pp. 38-54. Princeton University Press, 1924.

[32] A summary of this report may also be found in *Readings in Educational Psychology*, by Skinner, Gast and Skinner, pp. 655-658. D. Appleton and Co., 1926.

[33] The results of this study were submitted in the *Journal of Educational Psychology*, Vol. XVIII, pp. 377-404, September, 1927.

[34] For an abstract of this chapter see Skinner, Gast and Skinner, *op. cit.*, pp. 658-666.

4. *The amount of transfer in any case where transfer is admitted at all, is very largely dependent upon methods of teaching. This is probably the strongest note struck by the psychologists in their comments.*

5. A majority of the psychologists seem to believe that, with certain restrictions, *transfer of training is a valid aim in teaching.*

6. *Transfer is most evident with respect to general elements—ideas, attitudes, and ideals. These act in many instances as the carriers in transfer. Often they form the common element so generally held to be the sine qua non of transfer.*[55]

The Report then submits in detail the personal replies of these twenty-four psychologists, in alphabetic order, thus giving an interesting symposium on the present status of the mental discipline controversy. After a perusal of these opinions, it is no longer possible to justify the careless assertion that "mental discipline is a myth." This impression could be greatly strengthened by other significant quotations from current educational literature. We must limit ourselves to a few striking passages which may serve to dispel any latent doubts that may have remained in the minds of critical readers.

> *The fact of transfer cannot be doubted.* The factors involved in such transfer, the extent to which transfer can take place under any given set of conditions, and the best methods of securing such transfer will long doubtless remain questions for investigation and discussion.[56]
>
> The "disciplinary" function of systematic education is probably far more significant than is usually granted by the current interpretations of the experiments in "transfer of training." While it would be most unfortunate to go back to the naive conception of formal discipline that prevailed in the past, it would be the part of wisdom to go forward to a new conception which would aim to correct the unquestionable weaknesses, not to say, flabbiness of the position taken on this important issue by contemporary educational theory.[57]
>
> No one can doubt that all of the ordinary forms of home or school training have some influence upon mental traits in addition to the specific changes which they make in the particular function the improvement of which is their direct object. . . . The real question is not, "Does improvement of one function alter others?" but "To what extent, and how, does it?"[58]
>
> Special emphasis may be laid on the fact that *there is no one who denies that some kind of transfer takes place*. The real questions at issue are what is the degree of transfer and what is its method?[59]

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[55] *Report of the National Committee on Mathematical Requirements*, pp. 95-96.

[56] Colvin, S. S., *The Learning Process*, p. 241. Macmillan Co., 1917.

[57] Bagley, *op. cit.*, p. 160.

[58] Thorndike, E. L., *The Psychology of Learning*, p. 358. Teachers College, Columbia University, 1914.

[59] Judd, C. H., *Psychology of High School Subjects*, p. 404. Ginn and Co., 1915.

**The Extent of Transfer.** The earliest transfer experiments, because of obvious imperfections in the technique adopted, often yielded contradictory and non-conclusive results. Generally speaking, the evidence in favor of a liberal amount of positive transfer was not uniformly reassuring. The Classical Investigation made out a much more favorable case, though only in two very closely related fields of work, English and Latin. Thorndike's most recent studies, which are mentioned above, again stress the relatively meager amount of measurable transfer resulting from a study of the customary high school subjects. Thus Thorndike finds that "one year's study in either algebra or geometry *as now organized* does increase one's ability to think, *by a small degree*." He attributes the apparent superiority of Latin and mathematics to the fact that evidently the good students elect these subjects. "When the good thinkers studied Greek and Latin, these studies seemed to make good thinking." Hence, Thorndike believes that "after positive correlation of gain with initial ability is allowed for, the balance in favor of any study is certainly not large. Disciplinary values may be real and deserve weight in the curriculum, but the weights should be reasonable." [60]

We shall see below that there are excellent reasons why school subjects "*as now organized*" should lead to such a "small degree" of transfer. We do not secure transfer *unless we train for transfer*. In the meantime, the conclusions reached by Starch as to the extent of transfer, though dating back to 1919, are still sufficiently typical to warrant a restatement:

> 1. Practically every investigation shows that improvement in one mental or neural function is accompanied by a greater or less amount of modification in other functions.
>
> 2. This modification is in most instances a positive transfer, that is, an *improvement. Negative transfer,* that is, loss of efficiency in other functions, or interference, *has been reported principally among sensori-motor habits.*
>
> 3. The amount of improvement in the capacity trained is probably never accompanied by an equal amount of improvement in other capacities, with the possible exception of a few isolated instances whose actuality may be questioned.
>
> As a general estimate, on the basis of experimental work done thus far, the amount of transference between the extremes of 100% and 0% of transfer lies nearer to the zero end and is *probably in the neighborhood of 20% to 30%*

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[60] Thorndike, E. L., "Mental Discipline in High School Studies." *Journal of Educational Psychology*, Vol. XV, pp. 1-22, 83-98, January, 1924.

> *of transfer to closely allied functions* and from that point on down to 0% of transference to more unlike functions.
>
> 4. In the fourth place, *the improvement spread to other functions diminishes very rapidly in amount as these other functions become more and more unlike the function specifically trained.* This diminution occurs at a surprisingly rapid rate.[illegible]

Now, even if the actual, *measurable* transfer effect arising from the training of any mental function should consistently turn out to be small, it would still be necessary to guard against the error of underestimating the value of that small amount. This important fact is granted without hesitation by Professor Thorndike. To quote:

> It must be remembered that a very small spread of training may be of very great educational value if it extends over a wide enough field. If a hundred hours of training in being scientific about chemistry produced only one-hundredth as much improvement in being scientific about all sorts of facts, it would yet be a very remunerative educational force. If a gain of fifty percent in justice toward classmates in school affairs increased the general equitableness of a boy's behavior only one-tenth of one percent, this disciplinary effect would still perhaps be worth more than the specific habits.[illegible]

We may symbolize this view in a very simple manner. Suppose that $t$ denotes the hypothetical extent of transfer of a mental function. Let $a_1, a_2, a_3, a_n$, denote the number of potential opportunities of applying $t$ in various fields. Then the total transfer effect of $t$ may be indicated by the formula

$$T = t \ (a_1 + a_2 + a_3 + \dots\dots\dots + a_n).$$

Thus it appears that unless $t$ is negative or zero, the value of $T$ may be very considerable. These preliminary considerations will assume greater significance in the light of our subsequent discussion.

**Critical Evaluation of Current Experimental Procedures.** Whatever one's attitude may be with reference to the present findings of experimental psychology on the existence or the extent of transfer, it is impossible to deny or to overlook the obvious flaws in the prevailing experimental techniques and in the interpretation of alleged results.

> But very few experiments have been done with sufficient thoroughness and attention to scientific detail to merit the respect of an impartial investigator. As one reads the experimental literature, one seldom feels, with reference

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[illegible] Starch, *op. cit.*, p. 212.

[illegible] Thorndike, *Educational Psychology, Briefer Course*, p. 282.

> to any experiment, that it is final, that it settles that aspect of the question with which it deals. Few experimenters have repeated their experiments again and again, to see if every result confirmed every other. Too often the article reporting the experiments is only a "preliminary report." One usually searches the literature in vain to find a "full report" of the "main study." In more than one case, an experimenter has reported his results and given his inferences, while another psychologist would claim that different inferences were warranted from the results. Thorndike's inferences have been so questioned by Judd; Winch's, by Sleight. If educational psychologists are to command the respect of a scientific world, they must do their work with such thoroughness that it will stand the tests of repetition and criticism. Too often a class experiment that is scarcely worth anything as a mere demonstration is published as having scientific value.[43]

The whole question of technique is ably summed up by Professor J. W. A. Young, in the following interesting statements:

> The psychological researches that have been made are valuable, but they are as yet mere scratches on the surface of a field that needs to be mined deep. Mental acts at their simplest are complex. Good psychologists would be the first to tell us how little has yet been done in the direction of reaching an ultimate analysis of these complexes, or in the direction of determining to what extent the psychic powers at work in different mental acts are the same. . . . The results reached by psychologists working in the scientific spirit, when received and interpreted in the same spirit, can do only good. But danger lies in the accretions and distortions that the descriptions of these results undergo when handed down the line from science to rhetoric, from first to tenth hand. . . . The various analyses of the specialized and narrowly limited experiments that have been made agree with the statements just quoted; in the mathematical or laboratory sense of the word "proved" nothing has been proved one way or the other on general questions like that cited. *For such questions we still must be guided by the results of the experiences all down the centuries in the exacting laboratory of practical life; by myriads of practical experiences* like the following by one of the psychologists of the list (in the Report of the National Committee), experiences that are at least as significant (to put it mildly) as counting dots or crossing off *a*'s. W. C. Bagley "is convinced that students who come into his class in psychology after completing thorough courses in the higher mathematics do far better work than those who have not had this 'training.' Something has been carried over from one study to the other. It is certainly not the *habit* of study nor are the points that mathematics and psychology have in common sufficient to account for the difference."[44]

We shall not find it surprising, therefore, that there is a marked feeling among competent critics that the entire philosophy of mental testing is in need of much greater clarification[65] and that the tests themselves leave much to be desired. To quote:

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[43] Pyle, *op. cit.*, p. 215.

[44] Young, J. W. A., *The Teaching of Mathematics*, pp. 375 ff. Longmans, Green and Co., 1924.

> Our tests are at present totally inadequate for making the measurements sought for. Furthermore, I am absolutely sure that any such test is totally irrelevant on the practical side.[66]

> There is little general agreement on this whole topic, still less, if one means by transfer the effect of classroom work in any subject upon general ability or effectiveness in any other relation. All agree that habits may be trained and that many habits can be used in different connections; whether a child would reason better in politics because he had had a thorough training in mathematics, for example, is not proved by any actual work or measurements so far as I am familiar with the literature.[67]

> Our present tests are inadequate not only in testing specific habits which are learned, but also they do not in any way at present test general attitudes towards one's work or general methods of handling material and of thinking. It is my opinion that the higher the intelligence of the individual the more possibility of transfer, and the more this transfer will take place in the realm of general attitudes and methods. Stated in another way, *the less the intelligence of the child, the more emphasis must be placed on teaching in terms of transfer.*[68]

> The experimental facts so far determined are relatively trivial when compared with the facts which must be determined before we can secure measurements of the transfer or spread of improved efficiency in the case of school studies or the occupations of life. . . . The results of experimental investigations are confused and confusing, many being in direct conflict. Taken together, *they resist organization into any clear-cut statement of the extent of the effect of special training on general ability.* In addition, faulty methods employed in many cases quite invalidate the results obtained and the conclusions drawn. . . . Experiments have suggested that even where transfer or spread of improved efficiency is found, *its emergence is not automatic but dependent to a considerable extent on the methods employed in training.*[69]

Any candid reader of the literature on the experimental investigation of transfer is forced to admit that *"scientific methods," thus far, have not convincingly settled the question of the extent of transfer.* Nor have they thrown very much light, if any, *on the vitally important problem of how we can train effectively for transfer.* In other words, we are still in the predicament that was felt so keenly by Inglis more than ten years ago, as may best be stated in his own words:

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[65] See, especially, the excellent "Critique of Mental Measurements" by Thomas J. McCormack, in *School and Society*, Vol. XV, pp. 686-692, June, 1922.

[66] John B. Watson, as quoted in the *Report of the National Committee on Mathematical Requirements*, p. 103.

[67] W. B. Pillsbury, *ibid.*, p. 100. [68] Edward K. Strong, *ibid.*, p. 102.

[69] Inglis, *op. cit.*, pp. 408-409.

We may conclude that experimental evidence, while suggestive and indicative that older notions of general discipline are untenable, has as yet done relatively little to determine either the mode or extent of the transfer or spread of improved efficiency. We are therefore forced back on the field of general educational theory to a considerable extent. However unsatisfactory that may be, for the present we cannot do otherwise than consider its implications, with the hope and faith that improved methods of experimental psychology may soon afford more satisfactory evidence.[50]

## IV. PSYCHOLOGICAL THEORIES AS TO THE MODE OF TRANSFER

**The Range of Mental Functions.** Much of the uncertainty and confusion of thinking which is so apparent in the literature on the transfer of training is undoubtedly caused by our ignorance of the neural mechanisms which may be regarded as the physical foundation of mental phenomena. In spite of some progress in the field of neurology, even the most common of these phenomena, such as the operations of the memory, are largely shrouded in mystery. "We do not know exactly the way in which this process of retaining traces of past experience goes on. Some change in the minute molecular arrangement of nerve cells undoubtedly occurs."[51] The extreme complexity of even the "simplest" perceptual responses should be sufficient warning against too narrow an interpretation of any form of "specific" training. A better understanding of the mental cosmos may indeed change radically our entire conception of "specific" and "general" training. These considerations are of fundamental significance when we face the question of explaining *how* transfer takes place. The hierarchy of mental functions is so intricate that no theory of transfer will carry conviction which fails to give due attention to its many ramifications.[52] At the risk of some repetition, we shall therefore find it profitable to rehearse this fact more definitely by following McDougall's splendid exposition:

> There is in the structure of any mind something that endures as the ground of the potentiality of thinking of each specific object which can be thought of by that mind. . . . Perhaps the best term by which to describe it is *mental disposition;* for it is that which disposes or enables the mind to think of or to exercise its faculties, cognitive, affective, and conative, upon a corresponding object. . . . All minds of which we have any knowledge possess some dispositions, and the mind of every normal human adult possesses a vast

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[50] *Ibid.*, p. 409.

[51] Judd, C. H., in *The Nature of the World and of Man*, Chapter XVII.

[52] Reference may again be made to the work of Professor C. J. Herrick. See titles listed on page 158.

> number. The mind of a man is, in fact, a microcosm in which the world, in so far as he can be said to know it, is represented in detail, a disposition for every kind of object and every kind of relation of which he can think. If, for example, he can think of a horse, or a cube, or heat, or joy, or the causal relation, it is in virtue of the existence in his mind of a disposition corresponding to each of these objects.
>
> The many dispositions of any mind do not merely exist side by side; rather they must be conceived as functionally connected to form a vast and elaborately organized system; and this system is the structure of the mind. The more perfectly organized the mind, the more fully are the objects which compose the world and the relations between them represented in the mind by the dispositions and their functional relations. The total system formed by all the cognitive dispositions of the mind constitutes what is commonly called the knowledge possessed by that mind. . . . We have to conceive the cognitive dispositions as linked together in minor systems and these minor systems as linked in larger mental systems, and these again in still larger systems; and so on, by many steps of superordination, until the whole multitude are linked in the one vast system."[illegible]

The important thing in the simplest process of perceiving, says McDougall, is not merely the fact that "there is evoked in my consciousness a certain field of sensations of particular qualities and spatial arrangements." It is, rather, the *"meaning"* we attach to such a sensation.

> This *"meaning"* is the expression in consciousness of the coming into activity of a vast system of dispositions, built up in my mind through my thinking since the time I was a young child. . . . No disposition is an altogether new creation; every one arises rather as a specialization within some pre-existing disposition; and in this way, by the specializations within it of a number of minor dispositions, *a disposition becomes a system of dispositions.* And, when the constituent dispositions of such a system in turn become systems through the differentiation of new dispositions within them, the parent system becomes as it were a grandparent, and later by further similar steps a great-grandparent. The mental system may, then, be likened to a family the successive generations of which continue to live and work contemporaneously. . . . This is an over-simplified account of the growth and relation of mental systems."[illegible]

Another factor, which psychological mechanists have so often ignored, is the intricate relation of intellectual reactions to the *emotional* and *volitional* life of the individual.

> Hitherto we have considered the structure of the mind only in so far as it conditions cognition; but we have seen that all thinking is *affective* and *conative* as well as cognitive. And *knowing is but the servant of feeling and*

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[illegible] McDougall, W., *Psychology*, pp. 82-84. Henry Holt and Co., 1912.

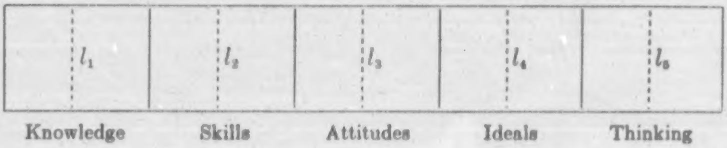
[illegible] *Ibid.*, pp. 86 ff.

> *acting;* it is the process by which the will works towards its end and the satisfaction which comes with the attainment of the end; and all the complex development of the conditions of the cognitive life . . . is achieved *through the efforts of the will to attain its ends.*"[illegible]

These considerations have tremendously important educational implications. Is it not true that we have viewed the curriculum in too narrow and therefore unproductive a sense? In the light of the "new" psychology, must we not regard a given school subject as a field of almost unlimited complexity, of countless interrelations, and therefore of endless "transfer" possibilities? Such an analysis leads us to the *theoretical* conclusion that *transfer is primarily a question of method.* Absence of transfer, after a period of "specific" or "general" training, is almost certain, then, to be a consequence of aimless or otherwise inefficient classroom procedures.

**The Doctrine of Identical Elements.** The view which explains the possibility of transfer solely on the basis of similar mental reactions or like neural reflexes has met with wide acceptance because it appears to offer a very plausible explanation of the mode of transfer.

Let us compare the potential richness of a school subject to the complete spectrum of ordinary sunlight. The "spectrum" of a subject is made up of innumerable elements, just as the spectrum of light contains innumerable "lines." In this analogy, the "primary colors" would correspond to such aspects as knowledge, skills, attitudes, appreciations, ideals, types of thinking, and the like.



Any *particular* skill, such as $l_2$, would signify a narrow strip of the spectrum (that is, a particular reaction mechanism of the brain). Now, let $S_1$ and $S_2$ represent the "spectra" of two different school subjects or types of activity. (See page 174.) In view of the close interrelations of the higher nervous mechanisms it is not only possible, but highly probable, that at some points—in the domains of knowledge, or of skills, or of attitudes, and so on—the two spectra

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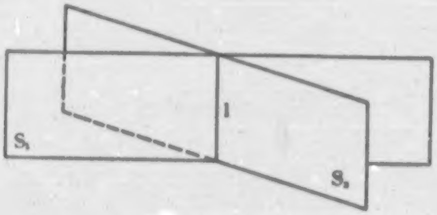
[illegible] *Ibid.*, pp. 104-105.

possess common lines such as *l* (that is, common neural bonds). In that case the stimulation of the common bond in *either* field of work would necessarily affect the *other*.



This theory has been endorsed especially by Thorndike. It is clearly described in the following statements:

> By identical elements are meant mental processes which have the same cell action in the brain as their physical correlate. It is of course often not possible to tell just what features of two mental abilities are thus identical. But, as we shall see, there is rarely much trouble in reaching an approximate decision in those cases where training is of practical importance.
>
> These identical elements may be in the *stuff*, the *data* concerned in the training, or in the *attitude*, the *method* taken with it. The former kind may be called *identities of substance*, and the latter, *identities of procedure*.
>
> The answer which I shall try to defend is that *a change in one function alters any other only in so far as the two functions have as factors identical elements.* The change in the second function is in amount that due to the change in the elements common to it and the first. The change is simply the necessary result upon the second function of the alteration of those of its factors which were elements of the first function, and so were altered by its training.[66]

Starch maintains that experimental data appear to be in harmony with this doctrine. He says,

> The evidence on spread of training in school material tends to support for the most part the theory of identical elements. The effects are the largest where there is similarity or identity of material, as, for example, in the case of the effect of the study of Latin upon the study of Spanish, or upon the knowledge of English grammar. The fact of identity of material or similarity of procedure makes possible a greater control of the spread of improvement through methods of teaching whereby the identity or the use of identical material may be emphasized in as many desirable relations as possible.[67]

[66] Thorndike, *The Psychology of Learning*, p. 358.
[67] Starch, *op. cit.*, p. 253.

A criticism of this theory must be postponed until we have examined in greater detail the rôle which identical elements seem to play as agencies of transfer.

**The Relation of Dissociation to Transfer of Training.** Suppose that a pupil is confronted with a number of situations that involve a common constant element *A* (knowledge, skill, attitude, and the like), but are otherwise made up of varying elements, as follows:

| | |
|---|---|
| (1) *A B C D* | (3) *A R S T* |
| (2) *A L M N* | (4) *A X Y Z* |

The presence of the identical element *A* in all these situations tends to impress itself upon the mind of the learner. The very fact that the other elements are all different *emphasizes* the element *A*. That is, the "*stimulus A*" is eventually felt as a distinct and separate stimulus. Hence it *dissociates* itself from the other "concomitant" stimuli. When that happens, the element *A* has been given a separate existence in the mind; it has been *generalized.* It is then capable of independent association with any suitable set of accompanying elements. That is, it can now be "transferred" to other situations. Thus Thorndike says, "Any element of mental life which is felt as a part of many total mental states, differing in all else save its presence, comes thereby to be felt as an idea by itself, and any movement which has been made as a part of many complex movements differing in all else save its presence comes thereby to be made as a movement by itself." [58]

"It is upon this process of dissociation," says Inglis, "that the abstraction of any general law, idea, principle, method, or the like must rest, and the process of developing abstract or general ideas is a process of dissociation. Since the law itself is but an expression of a mode of mental life which is innate it merely expresses the '*power of generalization*' which is innate in the human mind and must be considered as an original datum without which the growth of mental life would be impossible. The basis of the transfer or spread of improved efficiency is found in this law of dissociation or generalization." [59]

From the standpoint of transfer, then, the entire issue appears to hinge upon (1) the actual presence of *a common element* in a variety of situations; (2) the *dissociation* of this element by the learner. We shall see, however, that Thorndike's ideas require an important extension or precaution which has been most clearly formulated by Judd in his doctrine of generalization as the real basis of transfer.[60]

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[58] Thorndike, E. L., *Elements of Psychology*, p. 217. Teachers College, Columbia University, 2nd edition, 1907.

[59] Inglis, *op. cit.*, p. 398.

**The Doctrine of Generalization.** The history of science is full of instances showing that the common underlying cause of apparently unrelated phenomena was not observed until quite recently. The law of gravitation is a case in point. Similarly, a pupil may be confronted with situations exhibiting identical elements, without ever observing them. Hence Judd is led to formulate a theory of transfer which stresses the importance of a *conscious* recognition of the identical element, and the deliberate *search* for identical elements, as the basis of generalization. He says,

> When one studies the psychology of generalization he becomes aware of the uselessness of some of the formulas which have been proposed by those who hold that transfer of training takes place in cases where there are identical elements present. *The identical element is usually contributed by the generalizing mind.* On the other hand, there may be identical elements *potentially* present in various situations, but wholly unobserved by the untrained or lethargic mind. In fact, *the discovery of the identical element in a situation is in some cases the whole problem of training.*[61]
>
> In the same fashion we may show that the principles of intellectual economy which Thorndike frequently includes in his statement of identical modes of procedure, namely, the principles that one can learn to avoid distractions of all sorts, or that he can refuse to give up a piece of work even when it is uncomfortable, represent generalized identities of procedure which are not always realized. In all these cases we must distinguish sharply between the possibility of identical modes of procedure and the actual achieve-

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[60] A very careful distinction should be made between the rôle of the identical element *A* in any *specific learning process* and the function of *A* as an *agency of transfer*. Thus, if *A* is a conceptual element such as "five," the merely mechanical repetition of *A* does not guarantee an automatic mastery of its "meaning." Real comprehension or *understanding* of *A* is quite a different psychological problem, as Dewey pointed out in his acute criticism of the doctrine of identical elements. The development of "meaning" is a complicated process which involves real "thinking." "Thorndike is on solid ground in arguing for 'identities' of some sort, but this leaves us with the task of interpreting these identities." (Bode, B. H., *Modern Educational Theories*, p. 206. Macmillan Co., 1927.) The task of giving richness of "meaning" to *A* may actually involve many related mental functions and thus create a broad basis for transfer. This fact seems to constitute a sufficient refutation of the extreme doctrine of specific training, or "the law of specificity," as reiterated recently by followers of Pavlov. (See P. M. Symonds, "The Laws of Learning," in *The Journal of Educational Psychology*, Vol. XVIII, p. 409, September, 1927.) Hence the problem of transfer is inextricably "bound up with the problem of training in thinking." (Bode, *op. cit.*, p. 295.)

[61] Judd, *Psychology of High School Subjects*, p. 414.

ment of this identity. Such an achievement depends upon the exercise of trained intelligence. The existence of possible modes of procedure does not invariably lead to their realization in fact.[62]

**A Composite Theory.** Fortunately, the two theories presented above are not necessarily "diametrically opposite." Their supplementary nature is characterized by Inglis as follows:

> No two situations in life calling for action on the part of any individual are ever exactly alike in all respects. Hence training for an absolutely fixed and specific reaction to any given situation is an impossible and valueless process. *Strictly speaking there is no such thing as specific discipline.* Fortunately for the economy of mental life and efficiency in behavior it is possible for the mind to select certain parts of any total situation and react to those parts with a minimum of attention to other parts of the total situation. Since such parts of total situations may be essentially the same it is possible to establish what in all important respects are specific situation-response connections, and hence it is possible to assign values to specific discipline. However, through this same characteristic of the human mind comes also the possibility of abstracting from a number of total specific situations, differing with respect to most of their constituent elements, any given element which may be common to all the total situations or a majority of them. Thus we get the law of dissociation expressed by Thorndike.[63]

In any given situation, "whether or not dissociation or generalization takes place depends on two factors—the mental attitude or 'mind-set' which the individual brings to the situation, and the character of the situation experienced. *Subjective elements are no less important than objective elements.* It is perfectly possible for generalization to be potential in any set of situations without that generalization taking place because of the mind's attention to other elements than those involved in the dissociative element. On the other hand, it is perfectly possible for the mental attitude to project into objective situations a generalizing factor that is not highly fostered by the situation itself apart from subjective elements, though always there must be something in the objective situation to which the mind-set may be attached."[64]

Hence, following Inglis, we may formulate a composite theory of transfer that combines the views of Thorndike and of Judd. The essential factors which foster and facilitate transfer are now readily seen to be the following:

[62] *Ibid.*, p. 416.

[63] Inglis, *op. cit.*, pp. 397-398.

[64] *Ibid.*, p. 399.

1. A number of total situations must be experienced in which (*a*) the element to be dissociated and generalized is present in prominence, and (*b*) the other elements of the situation vary;
2. The element to be dissociated and generalized must be brought into the field of focal attention;
3. The element to be dissociated and generalized must be of such a character that it may be held in the mind as a separate element. This is commonly facilitated when a distinguishing name or other symbol may be attached, or when a generalized definition or law is formulated;
4. Practice must be given in applying the dissociated and generalized element in new situations.[65]

The theory outlined above may be summarized by the following key words: (1) *identical elements;* (2) *conscious dissociation;* (3) *generalization;* (4) *wide application.*

**A Transfer Formula.** The curriculum of a school offers many opportunities which may be used to foster transfer. Following Inglis, we may even symbolize "the possible extent and importance of the transfer or spread of improved efficiency" by a simple formula.[66] If we agree that the transfer value of the improved efficiency of any mental function is to be measured by "the sum total of its applications," we may let $a_1, a_2, a_3, \ldots \ldots a_n$ represent the *amounts of the transfer in various activities,* while $t_1, t_2, t_3, \ldots t_n$ represent the *number of occasions in each case.* Then the total transfer value in the case of any ability or habit may be measured by the formula

$$T = a_1t_1 + a_2t_2 + a_3t_3 + \ldots \ldots + a_nt_n.$$

This formula is illuminating. For it shows that in proportion as the number of *a's* or the number of *t's* is reduced, the value of $T$ also decreases. This fact is of the greatest educational significance, since it serves to give the reason for the usual meagerness of transfer as reported by most of the experimental studies mentioned above. *Actual* transfer and *possible* transfer are two very different things. This brings us to a consideration of the pedagogic aspects of the problem of transfer.[67]

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[65] Inglis, *op. cit.*, pp. 399-400.

[66] *Ibid.*, p. 406.

[67] Another interesting transfer formula has been suggested by Davis. This formula may be expressed as follows:

$$T = A \cdot E \cdot D \cdot K,$$

in which $T$ = the amount and value of the transfer; $A$ = the native ability of the learner; $E$ = the number of varied experiences of the learner; $D$ = the number of dissociations of common elements made from these experiences and then combined into new concepts or generalizations; and $K$, the number and kinds of ways in

## V. Educational Aspects of the Problem of Transfer

**Transfer Not Automatic.** The majority of psychologists at present are of the opinion that transfer is not automatic.[68] That is, there must be *conscious* attention in the classroom to the factors which favor transfer. Professor Judd has stated the case with admirable clearness:

> I do not think that any subject transfers automatically and in every case. The real problem of transfer is a problem of so organizing training that it will carry over in the minds of students into other fields. There is a method of teaching a subject so that it will transfer and there are other methods of teaching the subject so that the transfer will be very small. Mathematics as a subject cannot be described in my judgment as sure to transfer. All depends upon the way in which the subject is handled. . . . Transfer is a form of generalization, and training can be given so as to encourage generalization, or training can be of such a type as to hinder generalization.[69]

If this be correct, we must regard the teacher and the organization of the curriculum as crucial factors in the transfer problem.

> Since dissociation is the basis of the transfer or spread of improved efficiency and since the extensive operation of dissociation is fostered by these factors, it is clear that any subject of study which does not permit the organization of materials in teaching so as to meet the conditions suggested cannot be expected to offer the most favorable opportunities for transfer. Further, it is clear that, as far as indirect values are concerned, subjects of study may to some extent be measured according to the degree in which those conditions can be met. Moreover, since the method by which material is presented is also involved in meeting those conditions, it follows that transfer cannot be expected to operate most effectively, unless both subject-matter and the method of teaching are adapted to the conditions favoring the process of dissociation and generalization.[70]

**The Teacher and the Curriculum as Factors of Transfer.** The following five factors, according to Davis, are essentials of any

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which the new generalization is applied to specific situations or problems in life. (See Davis, *op. cit.*, pp. 137-138.)

Here, again, the value of $T$ is directly proportional to the factors $A$, $E$, $D$, $K$, thus giving us conclusions similar to those suggested above.

[68] *The Classical Investigation* reports (Part I, pp. 56 ff.) that of the fifty-nine psychologists who expressed an opinion as to the automatic nature of transfer, thirty-three stated that no automatic transfer occurs at all, or that it is slight or negligible. Only nine believed that transfer is to a *considerable extent* or almost entirely automatic. The remaining seventeen replies maintained an intermediate position or admitted a carefully qualified amount of automatic transfer.

[69] C. H. Judd, as quoted in the *Report of the National Committee on Mathematical Requirements*, pp. 99-100.

[70] Inglis, *op. cit.*, p. 400.

educational process designed to yield worth-while returns of a formal disciplinary sort:

> (1) Systematized knowledge; (2) likeness of the knowledge elements and the thought process entering into the content material of the several fields of study involved; (3) the native ability of the learner—particularly in respect to the ability to dissociate elements—to note likenesses and differences and to generalize from the data given; (4) the alertness and resourcefulness of the teacher in suggesting relationships and the possibilities of transfer; and (5) continued practice in the art of relating thought and action to different fields of interests.[11]

Douglas is convinced that "any teacher will be benefited by examining his courses of study and methods in the light of the theory of transfer of training . . . each subject should be so taught as to insure the greatest amount of transfer . . . *upon the teacher rests the chief responsibility for the indirect values received from his courses.*"[12]

> If the channels of transfer are identical elements of substance, method, and ideal, precautions must be taken to see that the course includes elements which will be found in out-of-school situations. The facts learned and the principles developed must be "identical" with the facts and principles encountered in daily work, in the duties of the home, or in social intercourse. Scrutiny of the course from this point of view will result in the elimination of material of little or no recognized social value, the rearrangement of certain topics, and the addition of others. Likewise, a degree of certainty and decision heretofore lacking with respect to the mental habits and ideals to be developed is demanded if these are to be influential in shaping instruction. Such an attitude will be of the greatest assistance in definitising and vitalising the instructor's daily work, for there can be no contradiction between the organization of a subject from the standpoint of identical elements and its arrangement on the basis of social values.
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> In order to insure maximum transfer, the teacher must not only have his immediate and ultimate aims of instruction clearly formulated, but he must see to it that his pupils share his views on the objectives to be gained. Many teachers never discuss with their pupils the reasons for which assigned work is given, nor say anything of the purposes or ends of their courses. The members of their classes are thus left to grope blindly for the meaning of it all, their feelings being but slightly relieved by the general statements which they hear to the effect that they are "training their minds," or that they should "take advantage of their wonderful opportunities to prepare for life."[13]

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[11] Davis, *op. cit.*, p. 133.

[12] Douglas, A. A., *Secondary Education*, p. 359. Houghton Mifflin Co., 1927.

[13] Douglas, *op. cit.*, pp. 360-361.

**Insufficiency of Specific Training.** The advocates of the doctrine of specific training appear to have overlooked the startling educational insufficiency of their position.

> Fortunately—or unfortunately—no two situations or experiences in life are ever exactly alike. Hence to attempt to train an individual to meet life's problems solely by means of fixed and unvarying responses would be an impossible and fruitless task. The number of needed responses would be infinite. Here, therefore, the educational theory which seeks to rest its case on a basis of absolutely unique and specific disciplines is reduced to absurdity. Time and energy never would permit a mortal being to prepare himself specifically to meet all conceivable situations in life. His only hope and salvation is to economize effort by making use of general disciplines.
>
> Happily, too, as has also been indicated above, most situations in life are shot through and through with common elements. It is possible to abstract some of these and to make them the elements for building up larger concepts and generalizations which in turn can be employed in new situations. In particular, these life experiences furnish specific common elements out of which may be formed general notions of many common everyday occurrences, standards of taste, ideals, interests, attitudes, methods of procedure, habits of work or play, and possibly other desirable achievements. To develop these is the aim and purpose of formal discipline; to utilize the generalized abilities in numerous fields of thought and action is the end sought in the transfer of these powers.[74]

**Method as a Vital Factor in Transfer.** We shall now agree that, fundamentally, "the effectiveness of formal discipline is dependent upon the pupil, the teacher, and the subject of instruction. Since obviously these are always variable forces, the extent and value of general training are exceedingly variable outcomes. It cannot, however, be repeated too often that the justification for formal discipline rests not in the *amount* of ability transferred but in the *range* of activities to which it is applied when transferred. Further, whether or not transference of any considerable value takes place is dependent upon the avenues of approach that are opened for the transmission of the acquired ability. These avenues do not become established by themselves. They exist, if at all, because of an external stimulus directed to their establishment and to the fact that they are persistently employed."[75]

These considerations are in agreement with Inglis' contention that "any attempt to dissociate materials from methods of experiencing them is erroneous and useless. From a pedagogical point of view the character of objective material is inextricably associated with the way in which it is experienced. Things, objects, ideals, and so on, have no meaning for the individual other than as he experiences them. To say that generalization and transfer have no importance with respect to the values of educative materials but are important with respect to method ignores the fact that the character of materials in part determines method of training and that methods of training in part determine the character of the materials as far as their effect on the individual is concerned. An antithesis of materials and methods is psychologically and pedagogically false."[78]

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[74] Davis, *op. cit.*, p. 135.

[75] *Ibid.*, p. 137.

**Habit and Thinking in Their Relation to Transfer.** One of the greatest sources of confusion in the whole mental discipline controversy may be traced to vagueness as to the meaning and the function of "*habit.*" A narrow or mechanistic interpretation of habit formation inevitably leads to a virtual denial of transfer. And classroom procedures which are based on such a psychology of learning will almost inevitably result in a corresponding meagerness of a measurable "spread" of training. A brief examination of this psychological question is therefore unavoidable at this point.

The "spectrum" of a school subject involves certain tangible elements such as *information* (*knowledge*), and *skills*, and certain far more subtle elements such as *attitudes*, *ideals*, and *types of thinking*. Let the diagram suggest these five aspects of the classroom work in any subject-matter field. Suppose that we picture in this manner the spectra of the various school subjects. They all have their *K*'s, their *S*'s, *A*'s, *I*'s, *T*'s, and so on. If the teacher stresses merely the *K* or the *S* elements of any subject, there is likely to be "transfer" only in that narrow range of the spectrum. A "mechanical" teacher gets only "mechanical" results. Above all,

| | | | | |
|---|---|---|---|---|
| $K$ | $S$ | $A$ | $I$ | $T$ |

| | | | | |
|---|---|---|---|---|
| $K_1$ | $S_1$ | $A_1$ | $I_1$ | $T_1$ |

First Subject

| | | | | |
|---|---|---|---|---|
| $K_2$ | $S_2$ | $A_2$ | $I_2$ | $T_2$ |

Second Subject

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[78] Inglis, *op. cit.*, p. 410.

habits which revolve around the *S* elements alone will not lead to an extensive "spread" of training.

It is not an exaggeration to say that wrong notions on the relation of habit formation to the higher aspects of learning constitute the most outstanding single cause of the prevalent disease of educational barrenness. Twenty years ago Dewey pointed a warning finger to the danger of overdoing the mechanical and automatic phases of school work. He said,

> In some educational dogmas and practices the very idea of training mind seems to be hopelessly confused with that of a drill which hardly touches *mind* at all—or touches it for the worse—since it is wholly taken up with training skill in external execution. This method reduces the "training" of human beings to the level of animal training. Practical skill, modes of effective technique, can be intelligently, non-mechanically *used*, only when intelligence has played a part in their *acquisition*.[17]

Dewey never ceases to emphasize the importance of concepts, "meaning," and understanding. In sharp contrast with Thorndike he rejects *in toto* the procedure which would base *meanings* on the technique of identical elements. To quote:

> Conceptions are not derived from a multitude of different definite objects by leaving out the qualities in which they differ and retaining those in which they agree. . . . *Conceptions are general because of their use and application, not because of their ingredients.* The view of the origin of conception in an impossible sort of analysis has as its counterpart the idea that the conception is made up out of all the like elements that remain after dissection of a number of individuals. Not so; the moment a meaning is gained, it is a working tool of further apprehensions, an instrument of understanding other things. Thereby the meaning is *extended* to cover them. *Generality resides in application to the comprehension of new cases, not in constituent parts.* A collection of traits left as the common residuum, the *caput mortuum*, of a million objects, would be merely a collection, an inventory or aggregate, not a *general idea;* a striking trait emphasized in any one experience which then served to help understand some one other experience, would become, in virtue of that service of application, in so far general. Synthesis is not a matter of mechanical addition, but of application of something discovered in one case to bring other cases into line.[18]

In his treatise on *Human Nature and Conduct* (1922), Dewey examines the function of habit in its relation to intelligence and to conduct. He shows that habit is "an essential element in thinking and not simply the ability to do things in the absence of thinking."

[17] Dewey, John, *How We Think*, p. 52. D. C. Heath and Co., 1910.

[18] *Ibid.*, pp. 127 ff.

These ideas have recently been elaborated with great force and clearness by Bode, whose searching analysis of this difficult subject deserves to be studied carefully by every progressive teacher. His exposition is a continuous commentary on the danger of applying blindly a dubious psychology of learning.[79]

Bode points out that learning is not merely "*analytic*," as Thorndike seems to believe, but also "*synthetic*." His views on the relation of habit and thinking to transfer may be inferred from the following quotations:

> Our task in teaching is not the emphasizing and isolating of connections which are *already present*, but the *construction of something that is new*. It is not just a matter of analysis, but of synthesis as well. . . . Habit formation in Thorndike's sense is relatively unimportant, whereas the cultivation of thinking as a creative process or as the reconstruction of old habits [illegible] of fundamental importance. A psychology which reduces all thinking to habit encourages teachers to put all the emphasis on the kind of re[illegible] which springs from rote learning. In terms of curriculum making it [illegible]sizes the selection and organization of material for the purpose of mechanical habit formation, to the neglect of selection and organization designed to promote thinking. Such a psychology is not an ally of democracy, but an enemy. . . . Thorndike shares with behaviorism the disposition to reduce all changes in behavior to mechanical habit.
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> The adaptation of a habit to a specific situation requires some sort of "sizing up" of the situation. . . . In proportion as conduct is directed by the *meanings* of things we say that it is intelligent. As our store of meanings increases, our habitual reactions are changed. . . . When the discovery of meanings calls for a special procedure of reflection and inquiry, it is called *thinking*. We think because our former habits of response are inadequate, and the results of thinking show themselves in the modification of our habits. *Thinking may be defined as a process of finding and testing meanings. . . . Habit formation and thinking are not contrasting processes,* as traditional psychology taught us to believe. . . . *Thinking is a process of remaking old habits and forming new ones.*
>
> A *concept* represents a wide range of possible behavior in condensed and concentrated form. . . . At this point our discussion of thinking trenches on the problem of the transfer of training. From our present standpoint this problem centers on the development of concepts. Mechanical habits, i.e., responses cultivated in isolation, do not seem to facilitate transfer, but may even provide obstacles to transfer. . . . But when our habits interpenetrate and form systems of response, which on higher levels grow into concepts, we get the flexibility and adaptability that we have in mind when we speak of transfer of training. This is simply to say that *transfer takes place through meanings*, or that *transfer of training is just another name for intelligence.*

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[79] See, especially, the following three treatises by B. H. Bode: *Modern Educational Theories*, Macmillan Co., 1927; *Fundamentals of Education*, Macmillan Co., 1921; *Conflicting Psychologies of Learning*, D. C. Heath and Co., 1929.

> The moral of all this is that if we devote ourselves to the proper development of concepts, transfer of training will cease from troubling. *We have had a problem of transfer because we have failed to develop concepts so as to give them proper usefulness outside of the classroom.* The fact that the problem of transfer is, in the first instance, a "school problem" raises the suspicion that we have this problem on our hands because of the cleavage between the school and the life outside of the school. In the world of everyday affairs we do not seem to be troubled so much by the problem of transfer. But a school subject which in Judd's language is "so organized that it rotates around its own center" does not carry over, and this calls for explanation. The remedy lies obviously in the reorganization of the curriculum and teaching method so as to remove the cleavage. The problem of transfer is symptomatic of a defect in our educational aims and ideals. If we can bring the school into right relations with the life outside of the school, the problem of transfer will take care of itself.
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> *If transfer is synonymous with intelligence, it is futile to inquire whether there is such a thing as transfer of training.* It is more to the point to consider why or how the application of old meanings to new situations is so limited. . . . The problem is how to secure a wider range of application for intelligence. . . . This problem is not disposed of by repudiating the whole idea of transfer and multiplying the subjects in the curriculum so as to minimize the need of transfer. No education, however extensive, is very profitable if it does not bestow the power to deal with new situations. In order to facilitate transfer, our first concern must be to improve the quality of the concepts that are developed. . . . In a word, the problem of transfer is bound up with the problem of training in thinking.[80]

**Subject Matter as Related to Transfer.** Only too often the work of the classroom is limited to a recital of factual material and to the cultivation of mechanical skills. We forget that this is but half the story. There will be little or no transfer, if we persistently ignore the basic considerations outlined above.

According to Judd, "the most promising subject in the curriculum can be turned into a formal and intellectually stagnant drill if it is presented by a teacher who has no breadth of outlook and no desire to teach pupils how to generalize experience. On the other hand, a teacher who has the ability to train pupils to look beyond particular facts and to see their relations and their broader meanings can stimulate thinking with any material of instruction that comes to his hand."[81]

In like manner, "when we concern ourselves with the cultivation of attitudes and ideals, we are confronted once more with the need of a guiding principle or social ideal. How should any given subject be taught? We can present the subject in either of two ways. We can treat it as though it were a separate and distinct thing, carefully fenced in from everything else. Our aim then will be to impart a certain expertness in handling the facts and laws or formulae within that field. This is the sort of training that produces the expert or the technician; and *such training, when rigidly adhered to, reduces education to a glorified bag of tricks.* Or, in contrast, we can recognize the fact that every subject in the curriculum is interwoven with life, and we can make it our aim to show its broader meaning for human experience. On this basis teaching takes on a very different character and aim. When mathematics is taught, not only so as to show the abstract relations of numbers, but also to reveal its bearing on practical affairs and on the great discoveries that have revolutionized our conceptions of the universe, it will lose its formal and technical character and become invested with vital interest."[82]

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[80] Bode, *Modern Educational Theories*, pp. 184 ff.

[81] Judd, *Psychology of Secondary Education*, p. 422.

It cannot be repeated too often, therefore, that "no subject of instruction *guarantees* mental training. Anyone who asserts that mathematics or Latin or science is a mind-trainer makes a superficial statement which is not in keeping with experience nor defensible in theory. These subjects may, if properly treated, be very useful in training the highest intellectual powers, but they cannot guarantee that fortunate result. . . . There is no guaranty in its content that any subject will give *general training* to the mind. The type of training which pupils receive is determined by the method of presentation and by the degree to which self-activity is induced rather than by content. It is not far from the truth to assert that any subject taught with a view to training pupils in methods of generalization is highly useful as a source of mental training and that any subject which emphasizes particular items of knowledge and does not stimulate generalization is educationally barren."[83]

We may conclude these comments by submitting the following corroborating quotations:

> Whether and to what extent habits, knowledge, ideals, and attitudes function in a new experience depends to a large extent upon their organization. . . . For us to profit from our experience with a phenomenon, *we must know its relations.* Truly knowing the phenomenon means knowing these relations. . . . The important thing about a fact is its meaning, its relation to the world

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[82] Bode, *op. cit.*, pp. 212-213.

[83] Judd, *op. cit.*, pp. 422 ff.

> of other facts. . . . Briefly, we are always to ask in the face of a new fact, *what does it mean?* What new light does it throw on my world? What are its consequences? What facts are related to it? In what generalization does it find a place? What are the uses to be made of it?[84]

> Application is, however, a most difficult mental process and needs to be learned just as the original principle itself has to be learned. . . . Those who opposed the doctrine of formal discipline by saying that the school subjects at the present time do not give a generalized training are undoubtedly criticizing not the human mind, but our methods of instruction. . . . The cultivation of this power of generalization is the most important achievement in the student's education. It will not come without special endeavor on the part of the student and on the part of the teacher.[85]

> Thinking is not like a sausage machine which reduces all materials indifferently to one marketable commodity, but it is a power of *following up and linking together* the specific suggestions that specific things arouse.[86]

> To make ideals and attitudes operative in all fields the teacher must give them exercise in at least several fields.[87]

**The Relative Educational Values of School Subjects.** After all, the problem of transfer is epitomized in the following practical question: "How does the education which the pupil receives in school affect his subsequent thinking and conduct?"[88] Hence the quarrel as to the relative superiority of this subject or that is largely beside the point.

> A harmonious correlation of subjects is a higher achievement than is a mere study of each without reference to the others. Correlation can be achieved only when instruction rises to the level of *teaching pupils how to generalize.* It is to be expected, therefore, that wherever subjects are taught at the *lower* levels of mental achievement, there will be competition between the various subjects and a corresponding waste of mental energy rather than training in generalization.[89]

Nevertheless, in view of the fact that transfer is not automatic, it behooves each teacher to have clear notions on the direct and indirect educational possibilities of each important type of school activity.[90] He will then realize that "it is not a matter of indifference from the standpoint of mental discipline whether the student elects, for example, Latin, mathematics, philosophy, and chemistry on the one hand, or short stories, elementary agriculture, woodshop, and bird study on the other." In other words, there is danger in the assumption that "any one study is as good as any other for mental discipline."[91]

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[84] Pyle, *op. cit.*, pp. 235 ff.

[85] Judd, *Psychology of High School Subjects*, pp. 422 ff.

[86] Dewey, *op. cit.*, p. 39.

[87] Thorndike, E. L., *The Principles of Teaching*, p. 246.

[88] Judd, *Psychology of Secondary Education*, p. 427.

[89] *Ibid.*, p. 425.

[90] For a valuable discussion of direct and indirect values, see Inglis, *op. cit.*, Chapter XI. A formula for measuring the direct educational value of a subject is given by Davis, *op. cit.*, p. 130.

Bagley and Colvin have emphasized the probable superiority of pure mathematics over applied mathematics, and of pure science over applied science. In a practical age, this is likely to be an unpopular view.

> In general, the great discoveries of science have been made by those who were primarily scientists; who were dominated in their actions by the scientific ideal. Applied science as a rule comes later than pure science. . . . On the other hand, the supporters of pure science sooner or later must feel the need of applying theory to practice. A science that has no human significance, that has no ultimate use, cannot exist in a world of human values.[92]

Pure science, says Colvin, is of greater disciplinary value, because —

> 1) through the facts which it presents, ideals of procedure and of truth may be developed which function in a wider human experience, greatly to the uplift of the race; 2) the content and method of pure science is such that it has a broader field of application than has applied science, and can function as an identical or similar element in more situations than can applied science; 3) the emotion which the pure seeking after truth arouses is higher and less likely to be deadened by other emotions than are the ideals of economic improvement and social betterment, which are the aims of an applied science. These latter are apt to conflict with each other and to obscure the greater issue.[93]

Educators must be ever watchful lest the curriculum become a conglomerate of unrelated and unproductive morsels.

> There are so many special interests that just now seem to be clamoring for recognition, practical, humanitarian, aesthetic, that our school programs are in danger of being overcrowded with a variety of subjects which cannot well take the place in point of mental training of those which have for years been firmly established in the curriculum. The very multiplicity of the subjects that have enriched our programs offers a distraction, and furnishes a training in dispersed rather than concentrated attention, a training which is not needed and should not be desired. The trend of popular opinion is such that the new must come in, and it is not maintained that this opinion is

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[91] Colvin, *op. cit.*, p. 247.
[92] *Ibid.*, pp. 248-249.
[93] *Ibid.*, pp. 249-250.

> not on the whole sound; but let us see to it that this new element is assigned its proper place and given its just value. In this time of rapid change we need sanity in educational doctrine and practice as scarcely ever before.[94]

In estimating the potential richness of a school subject, and its relative rank in the curriculum, the teacher should take into account the *entire* spectrum of the subject, its *K, S, A, I, T* elements; the range and the frequency of its possible applications within the classroom and in later life; its actual or potential appeal to the learner; its available resources as a type of learning or as an activity; and the like. When these factors are consciously realized and constantly kept in mind, it will appear that in the hierarchy of learning there are mountain peaks and valleys, giants and dwarfs, and that all things will contribute to the consummation of creating in the learner a broadly cultivated mind that is a source of happiness to himself and of benefit to others.

## VI. Geometry as a Field of Transfer

**A Challenging Situation.** We shall undoubtedly agree with Inglis that "it is futile and criminal to establish the study of secondary-school mathematics on the basis of extensive transfer values and then to fail to meet the conditions necessary if any extensive amount of transfer is to be accomplished."[95]

> The first condition for the successful transfer of improved efficiency is that the trait which it is desired to transfer be developed in connection with the content of the training study. In the general discussion of transfer values it was suggested that subjects of study differ in the degree in which favorable conditions are afforded for the exercise of the desired trait and that the transfer of improved efficiency is primarily conditioned by the character of the original training material. Secondary-school studies differ in the extent to which desirable mental traits may be exercised, in the fitness of the materials for purposes of manipulation in teaching, and in the character of the materials as already organized for teaching. In these three respects *mathematics possesses advantages over many subjects of study*. The materials of mathematics, ranging all the way from the simplest to the most complex, may be manipulated almost at will, thus permitting the arrangement of conditions most favorable to dissociation. The organization of materials in the field of mathematics has been determined from the start for purposes of teaching. With regard to the ready manipulation of materials for the purpose of fostering transfer values mathematics shares prominence with the language studies. *With regard to the certainty and accuracy of its*

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[94] Colvin, *op. cit.*, p. 250.

[95] Inglis, *op. cit.*, p. 499.

> *data it supersedes all other subjects.* With regard to the opportunity which it affords for the exercise of valuable mental traits most desirable to transfer, if possible, *it is equaled by few and surpassed by none of the other subjects in the program of the secondary school.*[illegible]

This frank admission of the superior transfer value of mathematics, when contrasted with the open hostility of many educators to all forms of mathematical training beyond the unavoidable rudiments, represents a stirring challenge that should no longer be met with a helpless silence. Since we are limiting our discussion to the case of geometry, we are therefore forced to consider carefully at least three major questions:

1. Precisely what are the "disciplinary" and "cultural" aims of mathematics, and—in particular—of geometry?
2. Can the materials of instruction which are characteristic of geometry be so organized as to give greater prominence to these disciplinary and cultural elements?
3. What are the principal difficulties which appear to have prevented a more general realization of the potential transfer value of geometric training, and how may these difficulties be overcome?

**The Disciplinary and Cultural Aims of Mathematics.** The National Committee on Mathematical Requirements gave considerable attention to a significant formulation of the disciplinary and cultural aims of secondary mathematics. We shall find it helpful to use the following passages of its Report as a basis for our subsequent discussion:

> *Disciplinary Aims.* We should include here those aims which relate to mental training, as distinguished from the acquisition of certain specific skills discussed in the preceding section. Such training involves the development of certain more or less general characteristics and the formulation of certain mental habits which, besides being directly applicable in the setting in which they are developed or formed, are expected to operate also in more or less closely related fields—that is, to "transfer" to other situations. . . .
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> In formulating the disciplinary aims of the study of mathematics the following should be mentioned:
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> 1. *The acquisition, in precise form, of those ideas or concepts in terms of which the quantitative thinking of the world is done.* Among these ideas and concepts may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general, etc.), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another.

[illegible] Inglis, *op. cit.*, pp. 496-497.

2. *The development of ability to think clearly in terms of such ideas and concepts.* This ability involves training in:

*a. Analysis* of a complex situation into simpler parts. This includes the recognition of essential factors and the rejection of the irrelevant.

*b.* The *recognition of logical relations* between interdependent factors and the understanding and, if possible, the expression of such relations in precise form.

*c. Generalization;* that is, the discovery and formulation of a general law and an understanding of its properties and applications.

3. *The acquisition of mental habits and attitudes which will make the above training effective in the life of the individual.* Among such habitual reactions are the following: a seeking for relations and their precise expression; an attitude of inquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory.

4. Many of these disciplinary aims are included in the broad sense of *the idea of relationship or dependence*—in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking," that is thinking in terms of and about relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics.

*Cultural Aims.* By cultural aims we mean those somewhat less tangible but none the less real and important intellectual, ethical, aesthetic or spiritual aims that are involved in *the development of appreciation and insight and the formation of ideals of perfection.* As will be at once apparent the realization of some of these aims must await the later stages of instruction, but some of them may and should operate at the very beginning.

More specifically we may mention the development or acquisition of:

1. *Appreciation of beauty in the geometrical forms* of nature, art, and industry.

2. *Ideals of perfection as to logical structure,* precision of statement and of thought, logical reasoning (as exemplified in the geometric demonstration), discrimination between the true and the false, etc.

3. *Appreciation of the power of mathematics*—of what Byron expressively called "the power of thought, the magic of the mind"—and the role that mathematics and abstract thinking, in general, have played in the development of civilization; in particular in science, in industry and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the infinite and of the permanence of laws in mathematics tends to establish.[77]

It is at once apparent that geometry has a prominent share in the realization of each of these aims which harmonize at every point with the educational theories presented previously. Together with the direct or practical values of geometry they constitute the educational platform on which geometry must rest its case as a school subject.

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[77] The *Report of the National Committee on Mathematical Requirements,* pp. 8-10.

**The Reorganization of the Course of Study in Geometry.** While a mere enumeration of the possible values of any school subject does not guarantee their realization in the classroom, a clear statement of the general and specific objectives of every phase of school work is the first step toward the achievement of worthwhile results. Once these objectives are formulated, adequate provision must be made for them in terms of classroom procedures and of time allotments. The course of study in geometry thus far has centered almost entirely on content and logical sequence. Only crude beginnings have been made in the direction of genuine motivation and appreciation from a standpoint of generalized experience. Too often geometric training has amounted to little more than a "bag of tricks." We have not kept in mind that "reasoning" must become a much more vital performance if it is to contribute to a "liberal" education.

> Logical method is more than the mere knowledge of valid types of reasoning and practice in the concentration of mind necessary to follow them. . . . More than this is wanted to make a good reasoner, or even to enlighten ordinary people with knowledge of what constitutes the essence of the art. The art of reasoning consists in getting hold of the subject at the right end, *of seizing on the few general ideas which illuminate the whole, and of persistently marshalling all subsidiary facts round them.* Nobody can be a good reasoner unless by constant practice he has realized the importance of *getting hold of the big ideas and of hanging on to them like grim death.*[36]

If geometry is to retain a respected place in the curriculum, its basic facts, concepts, skills, habits, attitudes, and the like must be consciously explored and developed in the direction of general training. This can be done, but it will take much time and effort on the part of progressive teachers.

Who can deny that the world in which we live is a museum of form? The concepts of equality, of congruence, similarity, and symmetry are implanted in the very nature of things. The "mass production" so characteristic of our machine age is based on these same ideas. Measurement has truly been called the "master art." Indirect measurement underlies the art of surveying and of navigation. Without it the depths of fathomless space would forever remain a closed domain, and astronomy would again become an impotent chronicle of displacements. Exact geometric construction is the very foundation of many practical activities. Our imposing skyscrapers, our bridges, tunnels, locomotives, automobiles, our floating ocean palaces, our airplanes, and many other miracles of modern science and engineering—all owe their existence in large measure to the ideas, skills, and modes of reasoning which are characteristic of geometry. The blueprint is the language of the trades, and the map is the universal passport of the traveler. Geometry is the joint creation of many races and climes, the common heritage of all mankind. Philosophers, artists, scientists, craftsmen,—all have contributed to its present structure. As a model of perfect thinking it has no equal. As the science of space it will endure forever.

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[36] Whitehead, A. N., *The Aims of Education and Other Essays*, pp. 127-128. Macmillan Co., 1929.

In short, at every turn geometry opens large horizons, invites generalization, and beckons the mind to new conquests. Have we not allowed the mere logician, the systematic "thinker," and the conventional recitation, to cloud the issue, to rob school geometry of its many ramifications, its inherent beauty and potential richness? Who can doubt that as soon as courses of study and classroom methods can be readjusted in harmony with the true rôle of geometry in the modern world, its legitimate place in the curriculum will any longer be questioned?

**Geometry as a Unique Laboratory of Thinking.** It has been asserted previously that geometry is a training field *par excellence* in the most basic types of thinking. These may be characterized as *postulational* thinking, *critical* thinking, and *consecutive* or *cumulative* thinking. The constant necessity, in geometry, of deriving the ultimate validity of an argument from its fundamental assumptions, the challenging opportunity of testing the correctness of every step, and the cumulative performance of building up an organically interwoven system of truths,—these and the related aspects of geometric "reasoning" constitute the distinctive glory of a subject which has fascinated a legion of enthusiastic admirers throughout the ages.

And yet, there arises the natural query whether all this training —granting its reality and importance for the average pupil—could not be obtained from a less forbidding background. Above all, would not these types of thinking be developed anyway, without specific effort, by any intelligent person, in the great "school of life"?

The thesis that we may *incidentally* acquire all the mental training we need by "random movements," until we finally have learned how to make the necessary "adaptations," seems contrary to the verdict of common sense and of everyday experience. According to Dewey,

> Up to a certain point, the ordinary conditions of life, natural and social, provide the conditions requisite for regulating the operations of inference. The necessities of life enforce a fundamental and persistent discipline for which the most cunningly devised artifices would be ineffective substitutes. The burnt child dreads the fire; the painful consequence emphasizes the need of correct inference much more than would learned discourse on the properties of heat. Social conditions also put a premium on correct inferring in matters where action based on valid thought is socially important. These sanctions of proper thinking may affect life itself, or at least a life reasonably free from perpetual discomfort. The signs of enemies, of shelter, of food, of the main social conditions, have to be correctly apprehended. But this disciplinary training, efficacious as it is within certain limits, does not carry us beyond a restricted boundary.[80]

The world's important work, for the most part, is done by men and women who have disciplined minds, who succeed in attacking and solving new problems by a persevering application of tested methods. Geometry does not guarantee, any more than any other subject, an automatic transmission of superior mental ability. There is no magic by which we may acquire such an equipment. But geometry is a unique laboratory of thinking, and as such it fosters the persistent and systematic cultivation of the mental habits which are so essential to all those who would claim mental independence and genuine initiative as their birthright.

A shallow appraisal of geometry is likely to emanate from an equally uninformed conception of the rôle of "thinking" in the economy of our daily routine. Such critics may be referred for corrective treatment to a careful perusal of Dewey's classic monograph on *How We Think*. We shall therefore close these fragmentary references to geometric types of thinking by a final quotation from that masterpiece of America's most noted educational philosopher:

> While it is not the business of education to prove every statement made, any more than to teach every possible item of information, it is its business

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[80] Dewey, *How We Think*, p. 20.

to *cultivate deep-seated and effective habits of discriminating tested beliefs from mere assertions, guesses, and opinions;* to develop a lively, sincere, and open-minded preference for conclusions that are properly grounded, and to ingrain into the individual's working habits methods of inquiry and reasoning appropriate to the various problems that present themselves. No matter how much an individual knows as a matter of hearsay and information, if he has not attitudes and habits of this sort, he is not intellectually educated. He lacks the rudiments of mental discipline. And since *these habits are not a gift of nature* (no matter how strong the aptitude for acquiring them); since, moreover, *the casual circumstances of the natural and social environment are not enough to compel their acquisition,* the main office of education is to supply conditions that make for their cultivation. The formation of these habits is the Training of Mind.[190]

**The Crucial Question: How can Geometric Training be Generalized?** To make available for *general* use the distinctive modes of thinking employed in geometry is now seen to be the most crucial problem of geometric instruction. As a rule, these types of thinking remain imbedded in a purely geometric background. They rarely emerge in the pupil's consciousness as tools available for use outside the field of geometry. This difficulty has been described with great clearness by Inglis. He says:

> Assume that it is desired through the study of geometry to develop a generalized method to be employed in the reflective thinking (reasoning) involved in problem solving—an element which is certainly involved in the processes of geometry and in every other field of mental activity. Call that element *A*. If we wish to facilitate the process of its dissociation it must be kept constant in the teaching of geometry. But also other elements in the total situations must be made to vary. It is here that difficulty arises, since it is extremely difficult to prevent certain other elements from remaining constant. Thus there is always present an element which makes it possible for us to recognize that we are dealing with geometry—certain concepts of space and number relations, and certain elements peculiar to the mathematics "class," classroom, or teacher. Some of those elements remain constant in spite of attempts to vary elements of specific content, exercises, problems, etc. Hence the normal situations in teaching geometry may be represented by such combinations of elements as
>
> (1) *A B C D E F*
> (2) *A B C G H I*
> (3) *A B C J K L*
> (4) *A B C M N O*
> (–) etc., etc.
>
> and as a result conditions favor not the dissociation of the desired element *A*, but the constant association of *ABC*. Thus in the great majority of cases

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[190] Dewey, *op. cit.*, pp. 27-28.

> the teaching of geometry in our secondary schools tends to favor, not the isolation and generalization of general methods of reflective thinking related to problem solving, but the close association of such methods to elements of geometrical content—a situation to some extent interfering with the process of transfer.[101]

We must not infer, however, that such dissociation and subsequent transfer is impossible.

> Any such conclusion would imply that all the individual's experiences in reflective thinking and problem-solving outside the geometry classroom are isolated from his experiences in that classroom, and would leave out of account or minimize the innate capacity of the mind to generalize on the basis of such other experiences—a capacity differing among individuals apparently according to original endowment. It would also leave out of account the possibility that the desired method, principle, or the like, may be isolated by the teacher or other individual and raised into consciousness in terms of a general law, rule, maxim, etc., expressed in terms which do not specifically associate content elements. Hence the bearing of Bagley's statement:[102]
>
> "Unless the ideal has been developed *consciously*, there can be no certainty that the power will be increased, no matter how intrinsically well the subject may have been mastered."[103]

In view of the difficulties pointed out above, it is not surprising that in the average classroom very little has been done thus far by way of generalizing the geometric types of thinking. But the issue can not longer be evaded if geometry is to remain a prescribed school subject. It is certainly an inviting field of work which, like any other virgin soil, needs to be cultivated for years before conclusive procedures can be recommended. In the meantime, a reassuring message from the able pen of Professor J. W. A. Young will serve to allay the fears of teachers who may shrink from a task which at first might appear to transcend their insight or their ability.

> If the teacher himself has clear concepts of method and ideals of procedure, and if with a thorough mastery of the subject matter he exemplifies these concepts and ideals concretely in his work, though without any explicit discussion of them, has he then done all that he can or should do to secure the maximum measure of assimilation of these concepts and ideals according to the native capacity of each pupil? If he aims consciously at exhibiting the mathematical thought processes clearly and effectively to the pupil,

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[101] Inglis, *op. cit.*, p. 402.

[102] Bagley, W. C., *The Educative Process*, p. 216. Macmillan Co., 1905.

[103] Inglis, *op. cit.*, p. 403.

> beginning with the simplest steps such as are used in psychological tests, and proceeding gradually to the more complex as the pupil's mastery of the simpler ones warrants, has he done his full duty? If in his teaching he tries to make the pupil attain as large a measure of mastery as possible of the subject's processes, results, and spirit, may he rest content in the belief that the methods of instruction which are most conducive to mastery of the subject itself are also the most favorable to the spread of its ideals and methods to other situations? Pending evidence to the contrary, I should be disposed to assume an *affirmative* answer to these questions as a working basis.[104]

## SUMMARY AND CONCLUSION

It has been suggested again and again, in the preceding pages, that *the problem of transfer of training is fundamentally one of good teaching.* In teaching, as in every other field of human endeavor, "*we can expect to reap no more than we sow,* for the law of compensation operates in the realm of mind no less than elsewhere. Results in mental training follow surely only upon the expenditure of definite and intelligent effort, but with this they seem everywhere commensurate." [105]

But whereas the "born" teacher may obtain excellent results by an *instinctive* application of the principles on which transfer depends, there is no doubt that in the average classroom the efficiency of the work could be greatly increased by a *conscious* cultivation of these cardinal principles. The following findings which have at last emerged from the mental discipline controversy might well be incorporated in the creed and the daily practice of every progressive teacher:

1. *Training for transfer is a worth-while aim of instruction; from the standpoint of life it is the most important aim.*
2. *Transfer is not automatic.* "We reap no more than we sow."
3. *Every type of "specific" training, if it is to rise above a purely mechanical level, should be used as a vehicle for generalized experience.*
4. *"The cultivation of thinking is the central concern of education."* [106]

Someone has defined "education" as that which remains after everything that we have learned in school is forgotten. There is much wisdom in that whimsical statement. Unless we keep in mind that the daily routine activities of the classroom can be justified only if they lead, however slowly, to a *permanent* enrichment of the lives entrusted to our care, we shall have failed in our educational stewardship.

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[104] Young, *op. cit.*, pp. 377-378.

[105] Ruediger, *op. cit.*, p. 116.

[106] Dewey, J., *Democracy and Education*, p. 179. Macmillan Co.

And so, the essence of our discussion is a more precise definition, and a glorification, of real teaching. Hence we may conclude our report by endorsing the stimulating and hopeful outlook which has recently been pictured so ably by Professor Whitehead:

> The study of the elements of mathematics, conceived in this spirit, would constitute a training in logical method, together with an acquisition of the precise ideas which lie at the base of the scientific and philosophical investigations of the universe. Would it be easy to continue the excellent reforms in mathematical instruction which this generation has already achieved, so as to include in the curriculum this wider and more philosophic spirit? Frankly, I think that this result would be very hard to achieve as the result of single individual efforts. For reasons which I have already briefly indicated, all reforms in education are very difficult to effect. But the continued pressure of combined effort, provided that the ideal is really present in the minds of the mass of teachers, can do much, and effect in the end surprising modification. Gradually the requisite books get written, still more gradually the examinations are reformed so as to give weight to the less technical aspects of the subject, and then *all recent experience has shown that the majority of teachers are only too ready to welcome any practicable means of rescuing the subject from the reproach of being a mechanical discipline.*[187]

[187] Whitehead, *op. cit.*, p. 135.

# SOME DESIRABLE CHARACTERISTICS IN A MODERN PLANE GEOMETRY TEXT[1]

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**Origin of Geometry Texts.** The difficulties which beset the writer of a satisfactory text in plane geometry can best be appreciated if we approach the problem from the historical point of view. Let us briefly review, then, some of the facts concerning the origins of our geometry texts.

Almost all the important theorems and constructions of our elementary plane geometry are found in the *Elements* of Euclid. Euclid lived about 300 B.C., and his contribution was that he gathered together all the essential facts of geometry which were known at his time and arranged them in logical system. That he was not a mere compiler of results obtained by others, however, is seen from the fact that he invented new proofs for theorems when his sequence made earlier proofs inapplicable.[2] His ideal, only imperfectly achieved to be sure, was to have all his propositions follow as necessary conclusions from certain axioms, postulates, and definitions explicitly stated in the beginning. Here originated the concept of a "hypothetico-deductive system" which was destined to play such a prominent rôle in modern mathematical thought.

The important place occupied by the *Elements* in the history of human thought is attested by the fact that it has gone through over one thousand printed editions since 1482.[3] Sir Thomas Heath says of it, "This wonderful book with all its imperfections, which indeed are slight enough when account is taken of the date at which it appeared, is and will doubtless remain the greatest mathematical textbook of all time. Scarcely any other book except the

[1] This chapter is substantially the same as an address given by the writer before the Third Annual Conference of Mathematics Teachers at the University of Iowa, October 12, 1928, under the title "Some Points Concerning the Selection of a Text in Plane Geometry," and before the Mathematics Section of the University of Illinois High School Conference, November 22, 1929, under the present title.

[2] Heath, T. L., *Greek Mathematics*, Vol. I, p. 357.

[3] Smith, D. E., *History of Mathematics*, Vol. I, p. 103.

Bible can have circulated more widely the world over, or been more edited and studied."[4]

The nature of Euclid's *Elements* is made clear by the following words of Professor David Eugene Smith: "He has no intuitive geometry as an introduction to the logical; he uses no algebra as such; he demonstrates the correctness of his constructions before using them, whereas we commonly assume the possibility of constructing figures and postpone our proofs relating to the constructions until we have a fair body of theorems; he does not fear to treat of incommensurable magnitudes in a perfectly logical manner; and he has no exercises of any kind."[5] The important point for us to note in connection with the discussion that follows is that here was a book meant for mature, highly intelligent men—a systematic treatise for scholars. How inevitable were the difficulties which must arise when such a book was made the basis for the instruction of youth!

**Early Tendencies in Teaching Geometry.** The teaching of geometry during the past two hundred years has exhibited two tendencies which have a bearing on our discussion. First, there was the tendency to place it earlier in the curriculum. In 1726 geometry was mentioned as a subject studied by Harvard seniors.[6] During the latter part of the eighteenth century Euclid was taught to lower classmen at Harvard and Yale. This downward trend has gone on until to-day we find geometry offered to sophomores and even freshmen in our secondary schools.

The second tendency, which was an outgrowth of the first, was to modify Euclid in such a way as to make it more acceptable to immature learners. This was to prove a long and arduous task, as might have been expected in the case of a book with such prestige and wealth of tradition back of it. Among teachers and textbook writers there were conservatives who regarded a rigid adherence to the original Euclid as a sacred duty, and for a long time they seemed to dominate the situation. Such conservatism led all too frequently to a tragic formalism of which we cite but one illustration taken from England. In 1901 Perry[7] "criticized

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[4] Heath, T. L., *loc. cit.*

[5] Smith, D. E., *op. cit.*, p. 106.

[6] *Final Report of the National Committee of Fifteen on Geometry Syllabus* (1912), p. 29. Hereinafter called simply "The Report." This valuable document is now out of print. The part we are using here is the historical section by Professor Cajori.

[7] The Report, p. 28.

Oxford because, for the pass degree there, two books of Euclid must be memorized, even including the lettering of the figures, no original exercises being required." Only in recent years has the movement for reform made genuine progress, and even now the task is not complete.

**Present Factors to be Considered.** Let us now turn to some factors connected with our present-day school situation which make the problem of securing a satisfactory text more urgent, and at the same time more difficult, than ever before.

First, there is the pupil himself. The number of pupils enrolled in the high schools to-day constitutes a much larger proportion of that part of the total population which is of high school age than it did a decade or so ago. This means that there are now present in our tenth grade classes many more students representing the lower levels of mental ability than was formerly the case. Most students of this type do not intend to go to college, and it is hard to interest them in a difficult subject like geometry, especially if the textbook presentation is of an abstract and formal character. It is not surprising, therefore, to hear the charge made that there is a greater percentage of failures in geometry than in any other subject in the curriculum.

The position of geometry as a school subject has also been imperiled by the more varied course of study in the modern school which has brought it into competition with many other subjects claiming large practical and utilitarian values. The opponents of the theory of transfer of training have challenged the right of geometry to a place in the school on the grounds that the abilities involved in its study (for example, the ability to think logically) do not carry over into other fields of thought. Fortunately, the consensus of competent opinion now takes a more favorable view of transfer. Studies such as that undertaken by Miss Vevia Blair[*] for the National Committee in Mathematical Requirements show (1) that transfer does exist, and (2) that the amount of transfer is largely dependent on the methods of teaching. The second of these findings has large implications for all those who teach geometry.

All recent attempts to adjust this most venerable school subject to modern school conditions were foreshadowed by the famous reform program recommended by Professor Felix Klein of Germany more than twenty years ago.[9] He urged (1) that the instruction should "emphasize the psychologic point of view which considers not only the subject matter but the pupil, and insists upon a very concrete presentation in the first stages of instruction followed by a gradual introduction of the logical element"; (2) that there should be "a better selection of material from the viewpoint of instruction as a whole"; (3) that there should be "a closer alignment with practical applications"; (4) that "the fusion of plane and solid geometry, and of arithmetic and geometry" should be encouraged.

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[*] Blair, Vevia, "The Present Status of Disciplinary Values in Education" in *The Reorganization of Mathematics in Secondary Education*, pp. 89-104.

How very modern all this sounds! Even to-day a committee sponsored by the Mathematical Association of America and the National Council of Mathematics Teachers is just beginning to work on the fourth point. In this regard the Europeans are far ahead of us.[10]

The two reports which have given direction and sanction to geometry reform in this country during the past several years are *The Reorganization of Mathematics in Secondary Schools, A Report by the National Committee on Mathematical Requirements* (1923), and a report of the College Entrance Examination Board on geometry requirements. Progressive mathematics teachers are by this time thoroughly familiar with the contents of these reports and they need not be discussed in detail here.

The final stage in carrying out a reform movement is reached when the recommended changes and improvements are embodied in textbooks for classroom use. Books written with this object in view are now appearing in considerable numbers. In addition to modifying the content and organization of their subject in conformity with the recommendations of the National Committee, recent authors are attempting to take advantage of the findings of educational psychology and the results of careful classroom experimentation by expert teachers in order to secure a more effective presentation of their material. Much has already been done to make the student's work in geometry more interesting and more profitable, and the outlook is bright for still further improvements.

[9] The Report, p. 18. The reference is to Klein's *Elementarmathematik vom hoheren Standpunkte aus. Theil II.; Geometrie.* pp. 435, 437.

[10] See the British Association Report on *The Teaching of Geometry in Schools.* G. Bell and Sons, London, 1925.

**Guide Needed in Evaluating Texts.** The teacher should find it helpful to have some sort of guide to aid him in evaluating texts in the light of the recent progress which has been made. It is hoped by the present writer that the outline which follows will be of some service in this way. In arranging this outline he has used the sources referred to above, and has borrowed some suggestions from rating scales for texts in subjects other than mathematics and also from a certain anonymous rating scale which has appeared in connection with the advertising for a well-known geometry.

The limitations of the outline will be obvious to any one who studies it. Different points in it are not all of the same value. There has been no attempt to weight these points according to their relative value so as to enable one to "score" a text. The outline is subjective. Different teachers using it would not all choose the same geometry text. Perhaps this is a good point, after all, because it will allow play for the personal preferences of teachers. Some of the points in the outline are difficult to apply because of their generality. Perhaps further analysis would make it possible to split some of these up into sub-points of greater definiteness and applicability. With these preliminary explanations the outline itself is offered for consideration.

### AN OUTLINE FOR JUDGING GEOMETRY TEXTS

I. Points pertaining to the book as a whole
    A. Recency
        1. When written. If a first edition, was it tested in the classroom before publication?
        2. When revised.
    B. Author or authors
        1. Educational experience
            a. Has he (or have they) taught pupils of the same age and experience as those for whom the book is intended?
            b. Has he taught other pupils than these?
            c. Other educational experience.
        2. Scholarship
            a. Degrees. When and where received.
            b. Books and articles written.
            c. Standing as a scholar.

- C. Conformity with requirements
    - 1. The Report of the National Committee
        - a. As to objectives.
        - b. As to content.
        - c. As to terminology.
    - 2. The College Entrance Examination Board's Report
    - 3. Local requirements
- D. Is the book sound from a mathematical point of view?
- E. Method of presenting the material
    - 1. Style of writing
        - a. Is the vocabulary simple?
        - b. Are the explanations clear?
        - c. Is the text addressed directly to the student?
    - 2. Is the presentation such as to maintain the student's interest?
        - a. Does the book attempt to awaken in the student the desire to make geometrical discoveries for himself?
        - b. Is the approach to new topics psychological, with concrete and intuitive work first?

II. Points pertaining to specific phases of the work
- A. The introductory course. This refers to all work up through the first three or four theorems of Book I.
    - 1. Motivation. Does the author attempt to interest the student from the very beginning by showing him something of the value of the study of geometry?
    - 2. Definitions
        - a. Are they restricted to those really needed?
        - b. Are they introduced gradually in connection with constructions and drawings rather than in large groups?
        - c. Are they correct and clear?
        - d. Does the author test the student's understanding of the definitions with questions and exercises?
    - 3. Constructions, drawings, and measurements. Work of this kind, especially the constructions, is very important because it fixes the fundamental concepts in the pupil's mind through concrete experience.

Some paper folding and cutting to fix the idea of congruence may be helpful. The following constructions have been used in introductory work:

  a. Constructing a perpendicular to a line through a point on the line, and also from an exterior point.
  b. Bisecting lines and angles.
  c. Constructing an angle equal to a given angle.
  d. Bisecting an arc of a circle.
  e. Trisecting a right angle.
  f. Inscribing a regular hexagon in a circle.
  g. Constructing a triangle congruent to a given triangle by making certain parts equal.

4. Is the transition from intuitive to demonstrative geometry made gradually and naturally?
5. Does the author show the need for a logical proof
    a. By means of optical illusions?
    b. By showing the inaccuracy of measurement?
    c. By showing the exactness and generality of a logical proof?
6. Does the author make clear the ideal nature of geometric things?

B. Theorems

1. Provision for individual differences. Does the author give a list of theorems for use in a minimum course?
2. Does the author help the student to master theorems
    a By giving general methods of attack applicable to all or to large groups of theorems?
    b. By asking preliminary questions before each theorem, leading the student to discover for himself the method of proof?
    c. By giving a plan of attack for each theorem?
3. Does the author challenge the pupil to attempt an independent proof?
4. Does the author compel the student to think when studying a theorem?
    a. By omitting some of the reasons or some of the statements in the theorems?

    - b. By merely indicating the method of proof in the case of some of the easier theorems and requiring the student to supply the details?
    - c. By asking questions about subtle points in the proof?
  5. Does the author make the proofs of the most difficult theorems as easy as possible
    - a. By explaining the difficult points with greater thoroughness?
    - b. By producing a simpler proof than most texts?
  6. Does the author ever follow up the important theorems with a discussion of their broader significance and their relations to other theorems?

C. Exercises
  1. Are the *oral* exercises numerous and well chosen? It seems desirable to introduce more simple exercises involving but one, two, or three steps with the lettered figures in the text.
  2. Provisions for individual differences
    - a. Are the written exercises grouped according to difficulty?
    - b. Are the difficult exercises marked?
    - c. Are any "hints" given for the difficult exercises?
  3. Is the selection of exercises based on the principle that many easy exercises are preferable to a few difficult ones?
  4. Is the selection of exercises such as to emphasize the basal theorems?
  5. Is there a sufficient number of good *numerical* exercises based on each of the following:
    - a. Similar triangles.
    - b. The Pythagorean theorem.
    - c. The areas of rectilinear figures.
    - d. The mensuration of the circle.
  6. Is there a sufficient number of exercises representing good practical applications?
  7. Are there comprehensive lists of review exercises and questions at appropriate intervals?